Reconstruction Algorithms in Compressive Sensing: An Overview

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Abstract. The theory Compressive Sensing (CS) has provided a new acquisition strategy and recovery with good in the image processing area. This theory guarantees to recover a signal with high probability from a reduced sampling rate below the Nyquist-Shannon limit. The problem of recovering the original signal from the samples consists in solving an optimization problem. This article presents an overview of reconstruction algorithms for sparse signal recovery in CS, these algorithms may be broadly divided into six types. We have provided a comprehensive survey of the numerous reconstruction algorithms in CS aiming to achieve computational efficiency.

Keywords: compressive sensing · reconstruction algorithms · signal recovery · image processing · sampling theorem

1 Introduction

In recent years, Compressive Sensing (CS) has attracted considerable attention in areas of applied mathematics, computer science, and electrical engineering by suggesting that it may be possible to surpass the traditional limits of sampling theory. CS is the theory of reconstructing large dimensional signals from a small number of measurements by taking advantage of the signal sparsity. CS builds upon the fundamental fact that we can represent many signals using only a few non-zero coefficients in a suitable basis or dictionary. CS has been widely used and implemented in many applications including computed tomography [9], wireless communication [40], image processing [8] and camera design [20].

Conventional approaches to sampling images use Shannon theorem, which requires signals to be sampled at a rate twice the maximum frequency. This criterion leads to larger storage and bandwidth requirements. Compressive Sensing (CS) is a novel sampling technique that removes the bottleneck imposed by Shannon’s theorem. This theory utilizes sparsity present in the images to recover it from fewer observations than the traditional methods. It joins the sampling and
compression steps and enables to reconstruct with the only fewer number of observations.

This property of compressive Sensing provides evident advantages over Nyquist-Shannon theorem. The image reconstruction algorithms with CS increase the efficiency of the overall algorithm in reconstructing the sparse signal. There are various algorithms available for recovery as shown in section 3.

2 Historical Background

In the area of engineering the sampling theorem of Nyquist-Shannon has a tremendous role e it can be used frequently only for band-limited signals otherwise it requires larger storage space and measurements for high-dimensional signals [33]. However, practically reconstruction is even possible with fewer measurements and compression is also needed before storage [3]. These requirements can be fulfilled with CS. The field of CS has gained enormous interest recently. It is basically developed by D. Donoho, E. Candes, Justin Romberg and T. Tao [1,11].

In the framework of CS, the signals probed are firstly assumed to be sparse or compressible in some basis [1,10,12,31,43]. Consider a complex-valued signal $x$ which itself may or may not be sparse in the canonical basis but is sparse or approximately sparse in an appropriate basis $\Psi$. That is,

$$x = \Psi \theta.$$  \hspace{1cm} (1)

where $\theta$ is sparse or approximately sparse. A central idea of the CS theory is about how a signal is acquired: the acquisition of signal $x$ of length $n$ is carried out by measuring $m$ projections of $x$ onto sensing vectors $\{\varphi_i^T, i = 1, 2, ..., m\}$: $y_i = \varphi_i^T x$ for $i = 1, 2, ..., m$. For sensing efficiency, we wish to collect a relatively much smaller number of measurements, that is, one requires that $m$ be considerably smaller than $n$ ($m \ll n$), hence the name CS. This data acquisition mechanism is at the core of a CS system that marks a fundamental departure from the conventional data acquisition compression transmission de-compression framework: the conventional framework collects a vast amount of data for acquiring a high-resolution signal, then essentially discard most of the data collected (in the $\Psi$ domain) in the compression stage, while in CS the data is measured in a compressed manner, and the much reduced amount of measurements are transmitted or stored economically, and every bit of the measurements are then utilized to recover the signal using reconstruction algorithms. The data acquisition process in CS framework is described by

$$y = \Phi x.$$  \hspace{1cm} (2)

According to Eq.(1) and Eq.(2) can be written as $y = \Phi \Psi \theta$ (the size of the sparsifying basis or sparse matrix $\Psi$ is $n \times n$). The figure1 illustrates the relationship between the variables. Typically with ($m < n$), the inverse problem...
is ill-posed [29]. However, the sparest solution of Eq. (2) can be obtained by solving the constrained optimization problem

\[
\begin{array}{ll}
\text{minimize} & \| \Theta \|_0; \quad \text{subject to} : \Phi \Psi \Theta = y.
\end{array}
\]  

(3)

where \( \| \Theta \|_0 \) is the \( l_0 \) norm defined as \( \| \Theta \|_0 = \sum_{i=1}^{n} | \Theta_i |^0 \) = number of nonzero components in \( \Theta \).

Unfortunately, it turns out that Eq. (3) is a problem of combinatorial complexity: finding solution of Eq. (3) requires enumerating subsets of the dictionary to identify the smallest subset that can represent signal \( x \), the complexity of such a subset search grows exponentially with the signal size \( n \) [10]. A key result in the CS theory is that if \( x \) is \( r \)-sparse, the waveforms in \( \{ \phi^T_i, i = 1, 2, ..., m \} \) are independent and identically distributed random waveforms, and the number of measurements, \( m \), satisfies the condition:

\[
m \geq c \cdot r \cdot \log(n/r),
\]  

(4)

where \( c \) is a small positive constant, then signal \( x \) can be reconstructed by solving the convex problem

\[
\begin{array}{ll}
\text{minimize} & \| \Theta \|_1; \quad \text{subject to} : \Phi \Psi \Theta = y,
\end{array}
\]  

(5)

where \( \| \Theta \|_1 = \sum_{i=1}^{n} | x_i | \) [1].

Fig. 1. (a) Compressive sensing measurement process with a random Gaussian measurement matrix \( \Phi \) and discrete cosine transform (DCT) matrix \( \Psi \). The vector of coefficients \( s \) is sparse with \( K=4 \). (b) Measurement process with \( \Theta = \Phi \Psi \). There are four columns that correspond to nonzero \( s_i \) coefficients, the measurement vector \( y \) is linear combination of these columns [1].

3 Reconstruction Algorithms

CS comprises a collection of methods of representing a signal on the basis of a limited number of measurements and then recovering the signal from these
measurements [35]. The problem of how to effectively recover the original signal from the compressed data plays a significant role in the CS framework. Currently, there exists several reconstruction algorithms which are defined either in the context of convex optimization, or greedy approaches, among them we can mention [1,6,10,12,35,37,42].

To present an overview of reconstruction algorithms for sparse signal recovery in compressive sensing, these algorithms may be broadly divided into six types as show in Fig.2.

**Fig. 2.** Compressive Sensing: Reconstruction Algorithms and their Classification adapted from [37].

### 3.1 Convex Relaxation

With the development of fast methods of Linear Programming in the eighties, the idea of convex relaxation became truly promising. It was put forward most enthusiastically and successfully by Donoho and his collaborators since the late eighties [35,39].

This class algorithms solve a convex optimization problem through linear programming [12] to obtain reconstruction. The number of measurements required for exact reconstruction is small but methods are computationally com-
plex. *Basis Pursuit* [14], *Basis Pursuit De-Noising* (BPDN) [14], *Least Absolute Shrinkage and Selection Operator* (LASSO) [41] and *Least Angle Regression* (LARS) [21] are some examples of such algorithms. *Basis Pursuit* is a principle for decomposing a signal into an "optimal" superposition of dictionary elements, where optimal means having the smallest $l_1$ norm of coefficients among all such decompositions.

*Basis Pursuit* has interesting relations to ideas in areas as diverse as ill-posed problems, abstract harmonic analysis, total variation denoising, and multiscale edge denoising. *Basis Pursuit* in highly overcomplete dictionaries leads to large-scale optimization problems. Such problems can be attacked successfully only because of recent advances in linear and quadratic programming by interior-point methods.

In the paper [41] *Lasso* ($l_1$) penalties are useful for fitting a wide variety of models. Newly developed computational algorithms allow application of these models to large data sets, exploiting sparsity for both statistical and computation gains. Interesting work on the lasso is being carried out in many fields, including statistics, engineering, mathematics and computer science. Recent works show matrix versions of signal recovery called $||M||_1$ Nuclear Norm minimization [38]. Instead of reconstructing $x$ from $\Theta x$, Nuclear Norm minimization tries to recover a low rank matrix $M$ from $\Theta x$. Since rank determines the order, dimension and complexity of the system, low rank matrices correspond to low order statistical models.

### 3.2 Non Convex Minimization Algorithms

Many practical problems of importance are non-convex, and most non-convex problems are hard (if not impossible) to solve exactly in a reasonable time. Hence the idea of using heuristic algorithms, which may or may not produce desired solutions.

In alternate minimization techniques, the optimization is carried out with some variables are held fixed in cyclical fashion and linearization techniques, in which the objectives and constraints are linearized (or approximated by a convex function). Other techniques include search algorithms (such as genetic algorithms), which rely on simple solution update rules to progress. There are many algorithm proposed in literature that use this technique like Focal Underdetermined System Solution (FOCUSS) [34], Iterative Re-weighted Least Squares [13], Sparse Bayesian Learning algorithms [44], Monte-Carlo based algorithms [27]. Non-convex optimization is mostly utilized in medical imaging tomography, network state inference, streaming data reduction.

### 3.3 Greedy Iterative Algorithm

Due to the fast reconstruction and low complexity of mathematical framework, a family of iterative greedy algorithms has been widely used in compressive sensing recently [19]. This class algorithms solve the reconstruction problem by finding the answer, step by step, in an iterative fashion.
The fast and accurate reconstruction algorithms has been the focus of the study of CS, they will be the key technologies for the application of CS. At present, the most important greedy algorithms include matching pursuit and gradient pursuit [18,19].

The idea is to select columns of $\Theta$ in a greedy fashion. At each iteration, the column of $\Theta$ that correlates most with is selected. Conversely, least square error is minimized in every iteration. Most used greedy algorithms are Matching Pursuit [32] and its derivative Orthogonal Matching Pursuits (OMP) [42] because of their low implementation cost and high speed of recovery. However, when the signal is not much sparse, recovery becomes costly. For such situations, improved versions of (OMP) have been devised like Regularized OMP [36], Stagewise OMP [18], Compressive Sampling Matching Pursuits (CoSaMP) [35], Subspace Pursuits [15], Gradient Pursuits [22] and Orthogonal Multiple Matching Pursuit [30].

3.4 Combinatorial / Sublinear Algorithms

This class of algorithms recovers sparse signal through group testing. They are extremely fast and efficient, as compared to convex relaxation or greedy algorithms but require specific pattern in the measurements, $\Phi$ needs to be sparse. Representative algorithms are Fourier Sampling Algorithm [24], Chaining Pursuit proper is an iterative algorithm [25], Heavy Hitters on Steroids (HHS) [26].

3.5 Iterative Thresholding Algorithms

Iterative approaches to CS recovery problem are faster than the convex optimization problems. For this class of algorithms, correct measurements are recovered by soft or hard thresholding [7], [16] starting from noisy measurements given the signal is sparse. The thresholding function depends upon number iterations and problem setup at hand.

The Iterative Hard Thresholding (IHT) algorithm for the first time was suggested by Blumensath and Davies for recovery in compressed Sensing scenario [7]. This algorithm can offer the theoretical guarantee with its implementation which can be shown in the particular one [23]. The basic idea of IHT is to chase a good candidate for the estimate of support set which fits the measurement. The IHT algorithm is an algorithm with a simple implementation.

Message Passing (MP) algorithms [17] are an important modification of iterative thresholding algorithms in which basic variables (messages) are associated with directed graph edges. A relevant graph in case of Compressive Sensing is the bipartite graph with $n$ nodes on one side (variable nodes) and $m$ nodes on the other side (the measurement nodes). This distributed approach has many advantages like low computational complexity and easy implementation in parallel or distributed manner. Expander Matching Pursuits [28], Sparse Matching Pursuits [5] and Sequential Sparse Matching Pursuits [4] are recently proposed algorithms in this domain that achieve near-linear recovery time while using $O(s \cdot \log(n/s))$ measurements only. Recently, proposed algorithm of Belief Propagation also falls in this category [2].
3.6 Bregman Iterative Algorithms

Bregman method is an iterative algorithm to solve certain convex optimization problems. These algorithms provide a simple and efficient way of solving \( l_1 \) minimization problem. [45] presents a new idea which gives exact solution of constrained problems by iteratively solving a sequence of unconstrained sub-problems generated by a Bregman iterative regularization scheme. When applied to CS problems, the iterative approach using Bregman distance regularization achieves reconstruction in four to six iterations [45]. The computational speed of these algorithms are particularly appealing compared to that available with other existing algorithms. There are various algorithms available for recovery.

In the Table 1, we have listed some reconstruction algorithms complexity measures for Compressive Sensing.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complexity</th>
<th>Minimum Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis Pursuit [14], [15]</td>
<td>( O(n^3) )</td>
<td>( O(s \log n) )</td>
</tr>
<tr>
<td>OMP [15], [36], [42]</td>
<td>( O(s , m , n) )</td>
<td>( O(s \log n) )</td>
</tr>
<tr>
<td>StOMP [18]</td>
<td>( O(n \log n) )</td>
<td>( O(n \log n) )</td>
</tr>
<tr>
<td>ROMP [35], [36]</td>
<td>( O(s , m , n) )</td>
<td>( O(s \log^2 n) )</td>
</tr>
<tr>
<td>CoSAMP [36]</td>
<td>( O(m , n) )</td>
<td>( O(s \log n) )</td>
</tr>
<tr>
<td>Subspace Pursuits [15]</td>
<td>( O(s , m , n) )</td>
<td>( O(s \log (n/s)) )</td>
</tr>
<tr>
<td>EMP [28]</td>
<td>( O(s \log (n/s)) )</td>
<td>( O(s \log (n/s)) )</td>
</tr>
<tr>
<td>SMP [5]</td>
<td>( O(s \log (n/s) \log R) )</td>
<td>( O(s \log (n/s)) )</td>
</tr>
<tr>
<td>Belief Propagation [2]</td>
<td>( O(n \log^* n) )</td>
<td>( O(s \log n) )</td>
</tr>
<tr>
<td>Chaining Pursuits [25]</td>
<td>( O(s \log^* n \log^* s) )</td>
<td>( O(s \log^2 n) )</td>
</tr>
<tr>
<td>HHS [26]</td>
<td>( O(s \text{polylog}(n)) )</td>
<td>( O(\text{poly}(s, \log n)) )</td>
</tr>
</tbody>
</table>

In paper [15] and [18], Basis Pursuit can reliably recover signals with \( n = 256 \) and sparsity level up to 35, from only 128 measurements. The reconstruction algorithms OMP and ROMP can only be reliable up to sparsity level of 19 for same \( n \) and \( m \). The performance of Basis Pursuit appears promising as compared to OMP derivatives from minimum measurements perspective.

4 Conclusion

Broadly speaking, the theory of Compressive Sensing sub-sample consists of a signal and then use a reconstruction algorithm based on optimization to rebuild it. This property of compressive Sensing provides evident advantages over Nyquist-Shannon theorem. The image reconstruction algorithms with CS increase the efficiency of the overall algorithm in reconstructing the sparse signal.
During the review process did the survey and identify six types of reconstruction algorithms classes. In this article, we have provided a comprehensive survey of the numerous reconstruction algorithms discusses the origin, purpose, scope and implementation of CS in image reconstruction and compares their complexity.

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References


