Moving Path Following Control of Constrained Underactuated Vehicles: A Nonlinear Model Predictive Control Approach*

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This paper addresses the design of a stabilizing continuous time sampled-data Nonlinear Model Predictive Control (NMPC) law to solve the Moving Path Following (MPF) motion control problem for constrained under-actuated robotic vehicles. In this scenario, the robotic vehicle is tasked to converge to a desired geometric path, expressed with respect to a moving frame of reference, while satisfying the actuation constraints. This control problem is addressed in the NMPC framework. Specifically, first a suboptimal Lyapunov-based nonlinear auxiliary control law is designed to solve the MPF problem. Then, the latter is used for the design of a suitable terminal set and terminal cost of the MPC controller to enforce closed-loop guarantees. Exploiting the properties of the auxiliary control law, we show that for a suitable selection of the input constraints, the terminal set can be removed, resulting in a global region of attraction of the proposed controller. Simulation results are provided to illustrate the proposed control strategy.

I. Introduction

The motion control problem of under-actuated vehicles has been extensively investigated, where the proposed control approaches can be broadly classified into Trajectory Tracking and Path Following (PF) schemes. Trajectory tracking schemes require the robotic vehicle to converge to a reference trajectory with temporal constraints. On the contrary, the Path Following scheme requires the robotic vehicle to follow a given geometric path at a desired speed without temporal specifications. A detailed discussion on the trajectory tracking and path following schemes can be found in [1, 2]. In a classical PF scheme, the reference geometric path is stationary or fixed with respect to the frame of reference. An interesting motion control problem arises when the frame of reference of the geometric path itself is moving or varies with respect to time. This leads to a generalized Path Following motion control problem termed as the Moving Path Following (MPF) problem introduced in [3–5]. The Moving Path Following problem arises in applications such as tracking a moving target [6, 7], autonomous landing of Unmanned Aerial Vehicle (UAV) or docking of an Autonomous Underwater Vehicle (AUV) on a moving platform [8], and target estimation and tracking problems where the robot tracks a moving target while performing observability based maneuvers in order to estimate the target position [9, 10]. The motion control problem that arises in such applications cannot be cast in a classical path following/trajectory tracking framework with stability and robustness guarantees, due to the implicit time constraints imposed by the moving target with possibly varying linear and angular velocities. Further, the problem is more involved when explicitly considering the state and input constraints of the system which is of significant practical importance.

A large body of literature use Lyapunov based methods to design the classical path following controllers and prove the stability properties of such controllers. See [1, 11] and references therein. A recent work in [3] proposed and solved the unconstrained MPF problem for fixed-wing UAVs and stability of such a controller was demonstrated using Lyapunov based arguments. These results however do not explicitly account for the system constraints and are therefore applicable in a limited region where the control inputs do not violate the system constraints. Model Predictive Control (MPC), owing to its ability to explicitly handle system constraints, has emerged as an attractive alternative for control design of constrained systems. The seminal work in the design of provably stable MPC for nonlinear systems, termed Nonlinear Model Predictive Control (NMPC), presented in [12, 13] has accelerated its application to constrained nonlinear systems. Motivated by these observations, this paper proposes the design of a provably stable, continuous time sampled-data NMPC for the MPF motion control problem for a *constrained* under-actuated robotic vehicle. To this end, we extend the result of a Nonlinear MPC scheme for Path Following control of constrained under-actuated

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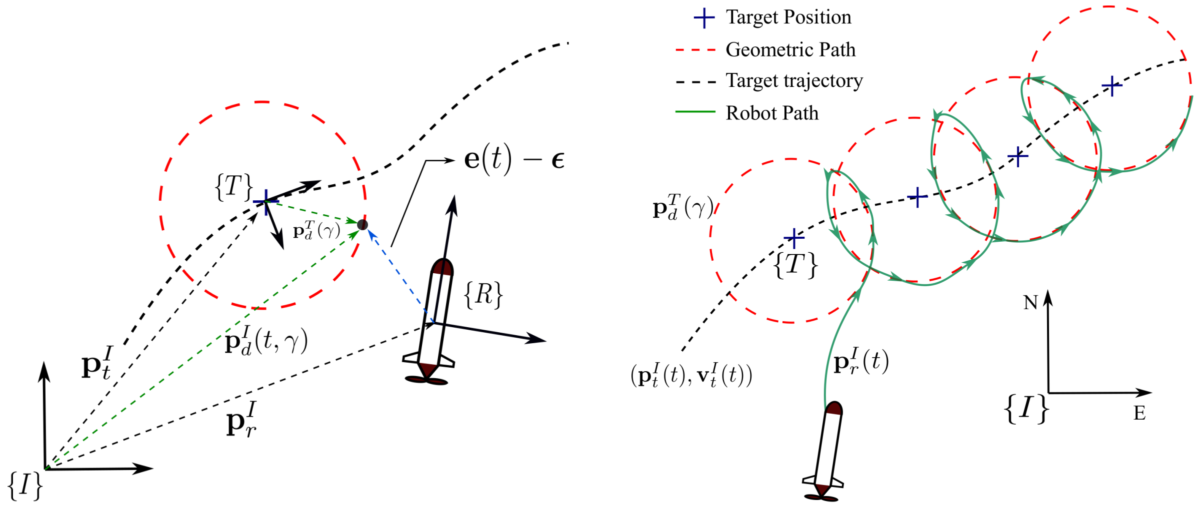


(a) Coordinate frames **(b) Circumnavigation example**

Fig. 1 Moving Path Following framework

vehicle proposed in [14] and adapt it to the constrained MPF problem. The constrained MPF motion control problem is formulated as an NMPC regulation problem using a conveniently defined error space and its stability is guaranteed through an appropriate design of terminal cost function and terminal constraint set. Motivated by the results of classical path following control, a Lyapunov-based nonlinear auxiliary control law is designed to solve the unconstrained MPF problem, that forms a crucial step towards the design of terminal cost and terminal constraint set. Furthermore, we show that the terminal constraint set can be excluded by choosing a bounded input constraint set over which a modified version of the auxiliary control law is always feasible. The results are validated through simulations where an under-actuated robot, modeled as unicycle kinematics is tracking a known target.

The remainder of the paper is organized as follows. The constrained MPF following problem is defined in Section II. Section III solves the unconstrained MPF control problem and presents the design of a Lyapunov-based nonlinear control law that forms a crucial result for the design of terminal cost function and terminal constraint set of the NMPC problem. The main result of design of a stabilizing continuous time sampled-data NMPC is presented in Section IV. Simulation results are presented in Section V followed by conclusions in Section VI.

II. Problem Definition

Consider an inertial reference frame $\{I\}$ and a target frame $\{T\}$ attached to a moving target with unknown dynamics. The position and the linear velocity of the target with respect to the inertial frame is denoted as $\mathbf{p}_t^I \in \mathbb{R}^2$ and $\mathbf{v}_t^I \in \mathbb{R}^2$, respectively, and is assumed to be known. Let $\mathbf{p}_d^T : \mathbb{R} \to \mathbb{R}^2$ be a fixed reference geometric path parameterized by $\gamma \in \mathbb{R}$ expressed in $\{T\}$ frame and $\dot{\gamma}_d \in \mathbb{R}$ be the desired speed assignment. The parameter γ could be the arc length along the reference path and the value of γ could be seen as a virtual point along the reference path. Then, the speed assignment $\dot{\gamma}_d$ dictates the evolution of the virtual point over time. Roughly speaking, the control objective is to make the robotic vehicle converge to vicinity of this virtual point. As an example, consider the scenario where a robotic vehicle is required to circumnavigate the moving target shown in Figure 1b and the associated coordinated frames in Figure 1a. Clearly, the reference geometric path moves along with the target at a velocity dictated by its motion. Notice that the reference geometric path imposes spatial constraints, while the temporal constraints are indirectly imposed by the moving target. Hence, such problems cannot be dealt in a classical path following framework and demands solution to a moving path following motion control problem.

The robotic vehicle is modeled using the kinematic model for an under-actuated vehicle given as,

$$\begin{aligned} \dot{\mathbf{p}}_r^I(t) &= R_R^I(t)\mathbf{v}_r^R(t) \\ \dot{R}_R^I(t) &= R_R^I(t)S(\omega_r) \end{aligned} \tag{1}$$

where $\mathbf{p}_r^I \in \mathbb{R}^2$ is the position of the robotic vehicle expressed in inertial frame, $\mathbf{v}_r^R(t) = [v_f(t) \;\; 0]^T$ is the linear velocity of the vehicle expressed in the robot body frame $\{R\}$ and $\omega_r(t)$ is the angular velocity of the robotic vehicle. The rotation matrix $R_R^I(t)$ represents the orientation of the robot frame $\{R\}$ with respect to the inertial frame of reference. The control inputs for the vehicle are defined as $\mathbf{u}(t) = [v_f \;\; \omega_r]^T$. Furthermore, let the control input $\mathbf{u}(t)$ be constrained to the set

$$\mathcal{U}(t) = \left\{\mathbf{u}(t) : v_{\min} \leq v_f \leq v_{\max}, \;\; \omega_{\min} \leq \omega_r \leq \omega_{\max}\right\} \tag{2}$$

Furthermore, the evolution of the path variable γ can be seen as an virtual control input, i.e, $\dot{\gamma}(t) = u_\gamma(t)$, constrained to the set

$$\mathcal{U}_\gamma(t) = \left\{u_\gamma(t) : u_{\gamma,\min} \leq u_\gamma \leq u_{\gamma,\max}\right\} \tag{3}$$

The MPF problem can be stated as follows.

Problem 1 (Constrained Moving Path Following). *Given a trajectory $\mathbf{p}_t^I(t)$ with time derivative $\mathbf{v}_t^I(t)$ and the desired geometric path $\mathbf{p}_d^T(\gamma)$ with desired speed $\dot{\gamma}_d$, the Moving Path Following control problem is to design a control law for $\mathbf{u}(t)$ that steers the vehicle along the desired moving path $\mathbf{p}_d^I(t,\gamma) = \mathbf{p}_t^I(t) + \mathbf{p}_d^T(\gamma)$ while satisfying the input constraints $\mathbf{u}(t) \in \mathcal{U}$. Specifically, we wish to drive the term $\|\mathbf{p}_r^I(t) - \mathbf{p}_d^I(t,\gamma)\|$ toward an arbitrarily small neighborhood of the origin as $t \to \infty$. Additionally, a virtual control input $u_\gamma(t) \in \mathcal{U}_\gamma(t)$ needs to be designed such that $\|u_\gamma(t) - \dot{\gamma}_d\| \to 0$ as $t \to \infty$.* □

III. Lyapunov-based Nonlinear Control Law

In this section, a globally exponentially stabilizing nonlinear control law is developed to solve the unconstrained MPF control problem. The existence of such a nonlinear control law is required to design a stabilizing nonlinear MPC law and forms an important intermediate result.

A. Error Dynamics

In order to design a globally exponentially stabilizing nonlinear controller using Lyapunov-based arguments, define an error variable $\mathbf{e}(t) = \left(R_R^I(t)\right)' \left(\mathbf{p}_r^I(t) - \mathbf{p}_d^I(t,\gamma)\right) + \boldsymbol{\epsilon}$. The vector $\boldsymbol{\epsilon} = [\epsilon_1 \;\; \epsilon_2]^T$ is a constant vector that can be made arbitrarily small. Taking the time derivative of the error variable $\mathbf{e}(t)$, we have

$$\dot{\mathbf{e}}(t) = \left(\dot{R}_R^I(t)\right)' \left(\mathbf{p}_r^I(t) - \mathbf{p}_d^I(t,\gamma)\right) + \left(R_R^I(t)\right)' \left(\dot{\mathbf{p}}_r^I(t) - \dot{\mathbf{p}}_d^I(t,\gamma)\right) \tag{4}$$

Notice that the desired position along the path $\mathbf{p}_d^I(t,\gamma)$ is expressed in the inertial frame which is not known. However, it can be computed from the known information as

$$\mathbf{p}_d^I(t,\gamma) = \mathbf{p}_t^I(t) + \mathbf{p}_d^T(\gamma) \tag{5}$$

Consequently, the time derivative of $\dot{\mathbf{p}}_d^I(t,\gamma)$ satisfies

$$\dot{\mathbf{p}}_d^I(t,\gamma) = \mathbf{v}_t^I(t) + \frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}(\dot{\gamma}) \tag{6}$$

Using the time derivative of $\dot{\mathbf{p}}_d^I(t,\gamma)$ and the robot dynamics (1) in (4) results in

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= \left(R_R^I(t) S(\omega_r)\right)' \left(\mathbf{p}_r^I(t) - \mathbf{p}_d^I(t,\gamma)\right) + \left(R_R^I(t)\right)' \left(R_R^I(t)\mathbf{v}_r^R(t) - \mathbf{v}_t^I(t) - \frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}(\dot{\gamma})\right) \\
&= -S(\omega_r)\left(\mathbf{e}(t) + \boldsymbol{\epsilon}\right) + \left(R_R^I(t)\right)' \left(R_R^I(t)\mathbf{v}_r^R(t) - \mathbf{v}_t^I(t) - \frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}(\dot{\gamma})\right) \\
&= -S(\omega_r)\mathbf{e}(t) + \Delta \mathbf{u}(t) - \left(R_R^I(t)\right)' \mathbf{v}_t^I(t) - \left(R_R^I(t)\right)' \frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}(\dot{\gamma})
\end{aligned} \tag{7}$$

where $\Delta = \begin{bmatrix} 1 & -\epsilon_2 \\ 0 & \epsilon_1 \end{bmatrix}$. In order to have a direct control over the angular velocity of the robotic vehicle we assume $\epsilon_2 \neq 0$.

B. Control law

Using the error dynamics of (7), the following result solves the unconstrained MPF control problem.

Proposition 1 (Unconstrained Moving Path Following)**.** *Consider the robotic vehicle* (1) *in closed loop with the feedback control law*

$$\mathbf{u}(t) = \Delta^{-1}\left(-K_p\mathbf{e}(t) + \left(R_R^I(t)\right)'\mathbf{v}_t^I(t) + \left(R_R^I(t)\right)'\frac{\partial \mathbf{p}_d^T(\gamma)}{\partial\gamma}\dot{\gamma}\right) \tag{8}$$

where K_p *is a known positive definite gain matrix and* $\boldsymbol{\epsilon}$ *chosen such that the matrix* Δ *in* (7) *is invertible. Then, the origin* $\mathbf{e}(t) = 0$ *of the closed loop system* (7) *with control law* (8) *is globally exponentially stable equilibrium point.*

Proof. Consider the ISS Lyapunov function

$$V(\mathbf{e}(t)) = \frac{1}{2}\mathbf{e}(t)'\mathbf{e}(t)$$

Taking the time derivative and using (7), (8), we have

$$\begin{aligned}\dot{V}(\mathbf{e}(t)) &= \mathbf{e}(t)'\dot{\mathbf{e}}(t)\\ &= \mathbf{e}(t)'\left(-S(\omega_r)\mathbf{e}(t) + \Delta\mathbf{u}(t) - \left(R_R^I(t)\right)'\mathbf{v}_t^I(t) - \left(R_R^I(t)\right)'\frac{\partial \mathbf{p}_d^T(\gamma)}{\partial\gamma}(\dot{\gamma})\right)\\ &= -\mathbf{e}(t)'S(\omega_r)\mathbf{e}(t) - \mathbf{e}(t)'K_p\mathbf{e}(t)\\ &\leq -\lambda_{\min}(K_p)\|\mathbf{e}(t)\|^2\end{aligned}$$

Consequently, the equilibrium point $\mathbf{e}(t) = 0$ of the system (7) with controller (8) is Globally Exponentially Stable. □

Remark 1. Note that the error $\|\mathbf{e}(t)\|$ converges exponentially to zero as $t \to \infty$ implies that the moving path following error $\|\mathbf{p}_r^I(t) - \mathbf{p}_d^I(t,\gamma)\| \to \|\boldsymbol{\epsilon}\|$ as $t \to \infty$. As a result, the robotic vehicle converges to an arbitrary small neighborhood of the origin.

IV. Nonlinear Model Predictive Control Design

A. Background

Among the main approaches used in the literature to certify the NMPC controller, we refer [12] for regional Lyapunov stability of nonlinear systems in the case of a quadratic stage cost and linear auxiliary control law valid around the origin. The results are extended in [13] and (possibly global) convergence to the origin using NMPC has been established through use of a nonlinear auxiliary control law. In this work, we use the recent result [15] to ascertain global asymptotic stability of the NMPC. Consider a generic nonlinear model of the system that needs to be stabilized using NMPC given as,

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{9}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, with $x(t_0) \in X_0 \subseteq \mathbb{R}^n$ the initial state at initial time t_0 and the control inputs $u(t) \in \mathbb{R}^m$ are constrained to the set $U : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^m$. The sampled-data Nonlinear Model Predictive Control problem is to solve an open-loop Optimal Control Problem (OCP) over a finite time horizon $T \in \mathbb{R}_{>0}$ at every sampling time instant $\mathcal{T} := \{t_i\}_{i\geq 0}$, with $t_{i+1} - t_i < T$, and $t_{i+1} = t_i + \delta$. The open-loop OCP is defined as follows,

Definition 1 (Open-loop OCP). Given a horizon length $T \in \mathbb{R}_{>0}$, let $\bar{x}([t_i, t_i + T])$ and $\bar{u}([t_i, t_i + T])$ denote the predicted state and input trajectories that satisfy the system dynamics with the initial condition $\bar{x}(t_i) = z(t_i)$. The open-loop MPC consists of solving the following optimization problem for the optimal control signal $\bar{u}^*([t_i, t_i + T])$.

$$\begin{aligned}\min_{\bar{u}^*([t_i,t_i+T])} \quad & \int_{t_i}^{t_i+T} l(s, \bar{x}(s), \bar{u}(s))ds + m(t_i + T, \bar{x}(t_i + T)) \\ \text{subject to} \quad & \dot{\bar{x}}(s) = f(s, \bar{x}(s), \bar{u}(s))\\ & \bar{x}(t_i) = z, \quad \bar{x}(t_i + T) \in \mathcal{X}_f\\ & \bar{x}(s) \in X, \quad \bar{u}(s) \in U\end{aligned} \tag{10}$$

where the variable $s \in [t_i, t_i + T]$ denotes the time variable used in the predictions, $l(s, \bar{x}(s), \bar{u}(s))$ is the recurring stage cost and $m(t_i + T, \bar{x}(t_i + T))$ denotes the terminal cost. The set $\mathcal{X}_f \subseteq \mathbb{R}^n$ denotes the terminal set. □

At any time instant $t_i \in \mathcal{T}$, the state of the system $x(t_i)$ is measured and used as an input to solve the OCP (10). From the computed optimal input trajectory $\bar{u}^*([t_i, t_i + T])$, only a part of the signal $\bar{u}^*([t_i, t_i + \delta])$ is applied to the system for a duration δ. At time t_{i+1}, the open-loop OCP is solved again with the new measurements (inputs) $z(t_{i+1}) = x(t_{i+1})$, thereby forming a feedback control loop. The time horizon, stage cost, terminal cost and the terminal set forms the design parameters for the NMPC problem. As will be discussed in the following, the terminal cost and terminal set plays a crucial role in design of stabilizing NMPC. At this point, we state the assumptions on the system and the sufficient conditions that needs to be satisfied for the design of a stable, sampled-data NMPC.

Assumption 1. The function $f(t, x, u)$ is locally Lipschitz with respect to x, piecewise continuous in t and u. Furthermore, the function is bounded for bounded x in region of interest i.e, set $\{\|f(t, x, u)\| : t \geq t_0, x \in X, u \in U\}$ is bounded for any bounded $x \in X \subset \mathbb{R}^n$ and $f(t, 0, 0) = 0$.

Assumption 2. The state constraint set $X(t)$ and the terminal set $\mathcal{X}_f \subseteq X(t)$ are closed, connected, and contain the origin for all $t \geq t_0$. The input constraint set $U(t)$ is such that $0 \in U(t)$ for all $t \geq t_0$.

Assumption 3. The running stage cost $l(s, \bar{x}, \bar{u})$ is continuous, $l(\cdot, 0, 0) = 0$ and there exists a class $\mathcal{K}_\infty$* function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $l(s, \bar{x}, \bar{u}) \geq \alpha(\|\bar{x}\|)$ for all $(t, \bar{u}, \bar{x}) \in \mathbb{R}_{\geq 0} \times U \times X(t)$. Moreover, for any given pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ the functions $l(t, \bar{x}, \bar{u})$ and $m(t, \bar{x})$ are uniformly bounded over time, with $m(\cdot)$ being positive semidefinite in x.

Assumption 4. There exists an auxiliary control law $k_{\text{aux}} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ such that the system (9) in closed-loop with the auxiliary control input $\bar{u}_{\text{aux}}(t) = k_{\text{aux}}(t, \bar{x}_{\text{aux}})$, with initial time and states $(\hat{t}, \hat{\bar{x}}) \in \mathbb{R}_{\geq t_i + T} \times \mathcal{X}_f$, the state and input trajectories exist, are unique, and respect the system constraints $\bar{x}_{\text{aux}} \in \mathcal{X}_f$, $\bar{u}_{\text{aux}} \in U(t)$ for all $t \in \mathbb{R}_{\geq \hat{t}}$. Here, $(\bar{x}_{\text{aux}}, \bar{u}_{\text{aux}})$ denotes the pair of state and input trajectories obtained by application of the auxiliary control law $k_{\text{aux}}(\cdot)$. Furthermore, for all $\bar{x}_{\text{aux}} \in \mathcal{X}_f$ and $\hat{t} \geq t_0$ the terminal cost satisfies,

$$m(\hat{t} + \delta, \bar{x}_{\text{aux}}(\hat{t} + \delta)) - m(\hat{t}, \hat{\bar{x}}) \leq -\int_{\hat{t}}^{\hat{t}+\delta} l(s, \bar{x}_{\text{aux}}, \bar{u}_{\text{aux}}) ds \tag{11}$$

Assumption 5. The time horizon T is chosen such that, the terminal set $\mathcal{X}_f$ is reachable from any initial state $x(t_0) \in X_0 \subset \mathbb{R}^n$ in time T. In other words, we assume that the OCP is feasible given the initial time and state pair $(t_0, x(t_0))$ such that the $x(t_0 + T) \in \mathcal{X}_f$.

Assumption 6. There exists a piecewise continuous control law $\bar{u}([\hat{t}, \hat{t} + T])$ for all $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^n$, such that the constrained system (9), in closed-loop with $\bar{u}([\hat{t}, \hat{t} + T])$ has feasible state and input trajectories and satisfies the inequality

$$\int_{\hat{t}}^{\hat{t}+T} l(s, \bar{x}(s), \bar{u}(s)) ds + m(\hat{t} + T, \bar{x}(\hat{t} + T)) \leq \alpha_c(\|\hat{x}\|) \tag{12}$$

for a class $\mathcal{K}_\infty$ function $\alpha_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

We now state the result of stabilizing MPC from [15]

Theorem 1. *Consider a system* (9) *and the design parameters of an MPC problem* $T, l(\cdot), m(\cdot), \mathcal{X}_f$ *that satisfies the assumptions 1 - 6. Then, for a sufficiently small inter-sampling time* δ*, the closed loop system resulting from application of the MPC strategy is asymptotically stable, i.e.,* $\|x(t)\| \to 0$ *as* $t \to \infty$.

Proof. Refer Theorem 1 of [15] for the proof. □

B. Main Result

Following the guidelines presented in the previous subsection, the solution of the constrained MPF problem using NMPC is presented in the following proposition.

*A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. Additionally it is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Proposition 2 (Constrained Moving Path Following)**.** *Given the error dynamics for the MPF system* (7)*, a finite horizon length T, a tuple of input parameters* $\mathbf{z}(t_i) = \left(\mathbf{p}_R^I(t_i), R_R^I(t_i), \gamma(t_i), \mathbf{p}_t^I(t_i), \mathbf{v}_t^I(t_i)\right)$*, consider the open-loop OCP*

$$\begin{aligned}
\min_{\bar{\mathbf{u}}([t_i,t_i+T])} \quad & \int_{t_i}^{t_i+T} \left(\|\bar{\mathbf{e}}(s)\|_Q^2 + \|\bar{\mathbf{u}}(s) - k_{\text{aux}}(s,\bar{\gamma},\bar{\mathbf{e}},R_R^I,\mathbf{p}_d^T,\mathbf{v}_t^I)\|_R^2 + \|\bar{u}_\gamma(s) - \dot{\gamma}_d\|^2\right) ds + m(\bar{\mathbf{e}}(t_i+T)) \qquad (13)\\
\text{subject to} \quad & \dot{\bar{p}}_r^I(s) = \bar{R}_R^I(s)\bar{\mathbf{v}}_r^R(s)\\
& \dot{\bar{R}}_R^I(s) = \bar{R}_R^I(s)S(\bar{\omega}_r(s))\\
& \dot{\bar{\gamma}}(s) = \bar{u}_\gamma(s)\\
& \bar{\mathbf{e}}(s) = \left(\bar{R}_R^I(s)\right)'\left(\bar{\mathbf{p}}_r^I(s) - \bar{\mathbf{p}}_d^I(s,\bar{\gamma})\right) + \boldsymbol{\epsilon}\\
& \bar{\mathbf{u}}(s) \in \mathcal{U}(s), \quad \bar{u}_\gamma(s) \in \mathcal{U}_\gamma, \quad \bar{\mathbf{e}}(t_i+T) \in \mathcal{E}_f\\
& \mathbf{z}(t_i) = \left(\mathbf{p}_R^I(t_i), R_R^I(t_i), \gamma(t_i), \mathbf{p}_t^I(t_i), \mathbf{v}_t^I(t_i)\right)
\end{aligned}$$

for all $s \in [t_i, t_i+T]$ *. The sampled data NMPC obtained by solving the OCP* (13) *at time* $t \in \mathcal{T}$ *and application of the computed optimal control input* $\mathbf{u}_{\text{mpc}}(t) = \bar{\mathbf{u}}^*([t_i, t_i+\delta))$ *for all* $t \in [t_i, t_i+\delta)$ *to the system* (1) *with the terminal cost ,*

$$m(t_i+T, \bar{\mathbf{e}}(t_i+T)) = \frac{\lambda_{\max}(Q)}{3\lambda_{\min}(K_p)}\|\bar{\mathbf{e}}(t_i+T)\|^3 \qquad (14)$$

and terminal constraint set $\mathcal{E}_f = \mathbb{R}^2$*, results in a stabilizing NMPC solution to the constrained MPF problem.*

Proof. The methodology adopted to prove the proposition is to motivate the choice of design parameters and show that the proposed design parameters satisfy the sufficient conditions for a stabilizing NMPC outlined in Theorem 1. For a suitable change of input, it is easy to see that the dynamical model (7) satisfies Assumption 1 considered with $x = e$.
Stage Cost – The most common stage cost function for tracking applications such the MPF problem is a quadratic stage cost. Consider the change of input coordinate $v = [\bar{\mathbf{u}}' - k_{\text{aux}}(s,\bar{\gamma},\bar{\mathbf{e}},R_R^I,\mathbf{p}_d^T,\mathbf{v}_t^I)', \bar{u}_\gamma - \dot{\gamma}_d]'$. Then, the cost function $l(s,\bar{\mathbf{e}},v) = \|\bar{\mathbf{e}}\|_Q^2 + \|\bar{\mathbf{u}} - k_{\text{aux}}(s,\bar{\gamma},\bar{\mathbf{e}},R_R^I,\mathbf{p}_d^T,\mathbf{v}_t^I)\|_R^2 + \|\bar{u}_\gamma - \dot{\gamma}_d\|^2$, for any $Q > 0$ and $R \geq 0$, trivially satisfies the conditions of the Assumption 3 with $v_{aux} = [k_{\text{aux}}(s,\bar{\gamma},\bar{\mathbf{e}},R_R^I,\mathbf{p}_d^T,\mathbf{v}_t^I)', \dot{\gamma}_d]'$.
Terminal Cost – In order to design an appropriate terminal cost, consider the error dynamics (7) with the Lyapunov-like function $W(\mathbf{e}) := \|\mathbf{e}\|$ presented in [16], and a modified version of the control input (8) defined as

$$\mathbf{u}(t) = \begin{cases} \Delta^{-1}\left(-K_p\frac{\mathbf{e}(t)}{\|\mathbf{e}(t)\|} + \left(R_R^I(t)\right)'\mathbf{v}_t^I(t) + \left(R_R^I(t)\right)'\frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}\dot{\gamma}_d\right), & \|\mathbf{e}(t)\| \neq 0\\ \Delta^{-1}\left(\left(R_R^I(t)\right)'\mathbf{v}_t^I(t) + \left(R_R^I(t)\right)'\frac{\partial \mathbf{p}_d^T(\gamma)}{\partial \gamma}\dot{\gamma}_d\right), & \|\mathbf{e}(t)\| = 0 \end{cases} \qquad (15)$$

The control input (15) in closed loop with (7) results in

$$\dot{\mathbf{e}}(t) = \begin{cases} -S(\omega_r)\mathbf{e}(t) - K_p\frac{\mathbf{e}(t)}{\|\mathbf{e}(t)\|}, & \|\mathbf{e}(t)\| \neq 0\\ 0, & \|\mathbf{e}(t)\| = 0 \end{cases}. \qquad (16)$$

The time derivative of the Lyapunov-like function $W(\mathbf{e}(t))$ takes the form

$$\dot{W} = \begin{cases} \frac{\mathbf{e}(t)'\dot{\mathbf{e}}}{\|\mathbf{e}(t)\|} = \frac{-\mathbf{e}(t)'K_p\mathbf{e}(t)}{\|\mathbf{e}(t)\|^2} \leq -\lambda_{\min}(K_p), & \|\mathbf{e}(t)\| \neq 0\\ 0, & \|\mathbf{e}(t)\| = 0 \end{cases} \qquad (17)$$

Consequently, the solution of the error variable $\mathbf{e}(t)$ converges to the origin in finite time as follows

$$\|\mathbf{e}(\tau)\| = \begin{cases} \|\mathbf{e}(t)\| - \lambda_{\min}(K_p)(\tau - t), & \tau \in [t, t_O]\\ 0, & \tau > t_O \end{cases} \qquad (18)$$

with $t_O := t + \frac{\|\mathbf{e}(t)\|}{\lambda_{\min}(K_p)}$. Let $(\mathbf{e}_{\text{aux}}, \mathbf{u}_{\text{aux}})$ denote the state and input trajectories obtained by application of the control law (15), with initial time and initial state denoted by $(\hat{t}, \hat{\mathbf{e}}) \in \mathbb{R}_{\geq t_i+T} \times \mathcal{E}_f$ for all $\tau \in [t_i+T, \infty)$. Then, the running stage cost can be upper bounded as

$$l(\tau, \mathbf{e}_{\text{aux}}, \mathbf{u}_{\text{aux}}) \leq \hat{l}(\tau; \hat{t}, \hat{\mathbf{e}}) = \begin{cases} \lambda_{\max}(Q)\left(\|\hat{\mathbf{e}}\| - \lambda_{\min}(K_p)(\tau - \hat{t})\right)^2, & \tau \leq \hat{t} + \frac{\|\hat{\mathbf{e}}\|}{\lambda_{\min}(K_p)}\\ 0, & \tau > \hat{t} + \frac{\|\hat{\mathbf{e}}\|}{\lambda_{\min}(K_p)} \end{cases} \qquad (19)$$

where $\hat{l}(\tau;\hat{t},\hat{\mathbf{e}})$ satisfies

$$\hat{l}(\tau;\hat{t}+\delta,\mathbf{e}_{\text{aux}}(\hat{t}+\delta)) \le \hat{l}(\tau;\hat{t},\hat{\mathbf{e}})$$
$$\lim_{\tau\to\infty}\hat{l}(\tau;\hat{t},\hat{\mathbf{e}}) = 0$$

for all $\delta \ge 0$. Invoking the result of Lemma 24 from [17], the terminal cost function

$$\begin{aligned} m(\hat{t},\hat{\mathbf{e}}) &= \int_{\hat{t}}^{\infty}\hat{l}(\tau;\hat{t},\hat{\mathbf{e}})d\tau \\ &= -\lambda_{\max}(Q)\frac{\left(\|\hat{\mathbf{e}}\| - \lambda_{\min}(K_p)\tau\right)^3}{3\lambda_{\min}(K_p)}\Bigg|_0^{\frac{\|\hat{\mathbf{e}}\|}{\lambda_{\min}(K_p)}} \\ &= \frac{\lambda_{\max}(Q)}{3\lambda_{\min}(K_p)}\|\hat{\mathbf{e}}\|^3 \end{aligned} \tag{20}$$

satisfies the cost decrease condition (11) of Assumption 4.

Terminal Constraint Set – In the following, an input constraint set is selected such that the auxiliary control law is always feasible and the terminal set can be omitted or equivalently, $\mathcal{E}_f = \mathbb{R}^2$. From (15), we have

$$\|[\mathbf{u}_{\text{aux}}(\tau)]_1\| \le \|[\Delta^{-1}]_1\|\eta + \|[\Delta^{-1}K_p]_1\| =: v_{\max} \tag{21}$$

$$\|[\mathbf{u}_{\text{aux}}(\tau)]_2\| \le \|[\Delta^{-1}]_2\|\eta + \|[\Delta^{-1}K_p]_2\| =: \omega_{\max} \tag{22}$$

where

$$\eta := \|\mathbf{v}_t^I\| + \sup_{\gamma}\left\|\frac{\partial \mathbf{p}_d^T(\gamma)}{\partial\gamma}\right\|\|\dot{\gamma}_d\| \tag{23}$$

choosing $v_{\min} = -v_{\max}$ and $\omega_{\min} = -\omega_{\max}$, leads to an input constraint set $\mathcal{U}$ that satisfies Assumption 2 along with the terminal set $\mathcal{E}_f$. The chosen stage cost, terminal cost, terminal set and the input constraint set satisfies Assumptions 1 - 5, leading to a stabilizing NMPC scheme for the MPF problem.

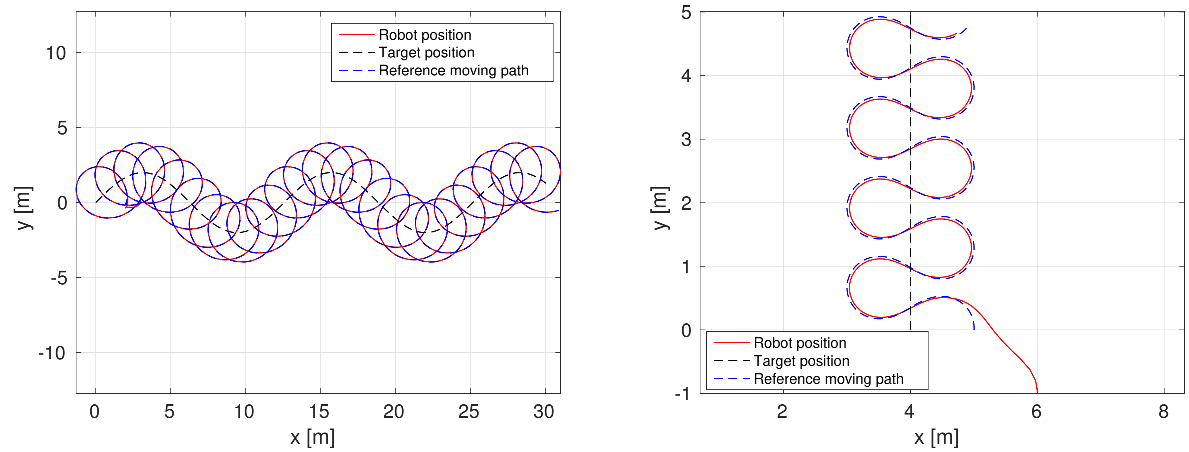
Controllability – The controllability assumption 6 is satisfied by noticing that

$$\begin{aligned} &\int_{\hat{t}}^{\hat{t}+T} l(s,\bar{x}(s),\bar{u}(s))ds + m(\hat{t}+T,\bar{x}(\hat{t}+T)) \\ &= \int_{\hat{t}}^{\hat{t}+T} l(s,\bar{x}(s),\bar{u}(s))ds + \int_{\hat{t}+T}^{\infty}\hat{l}(s;\hat{t}+T,\bar{x}(\hat{t}+T))ds \\ &\le \int_{\hat{t}}^{\hat{t}+T}\hat{l}(s;\hat{t},\hat{x})ds + \int_{\hat{t}+T}^{\infty}\hat{l}(s;\hat{t}+T,\bar{x}(\hat{t}+T))ds \\ &\le \int_{\hat{t}}^{\infty}\hat{l}(s;\hat{t},\hat{x})ds = \frac{\lambda_{\max}(Q)}{3\lambda_{\min}(K_p)}\|\hat{\mathbf{e}}\|^3. \end{aligned}$$

□

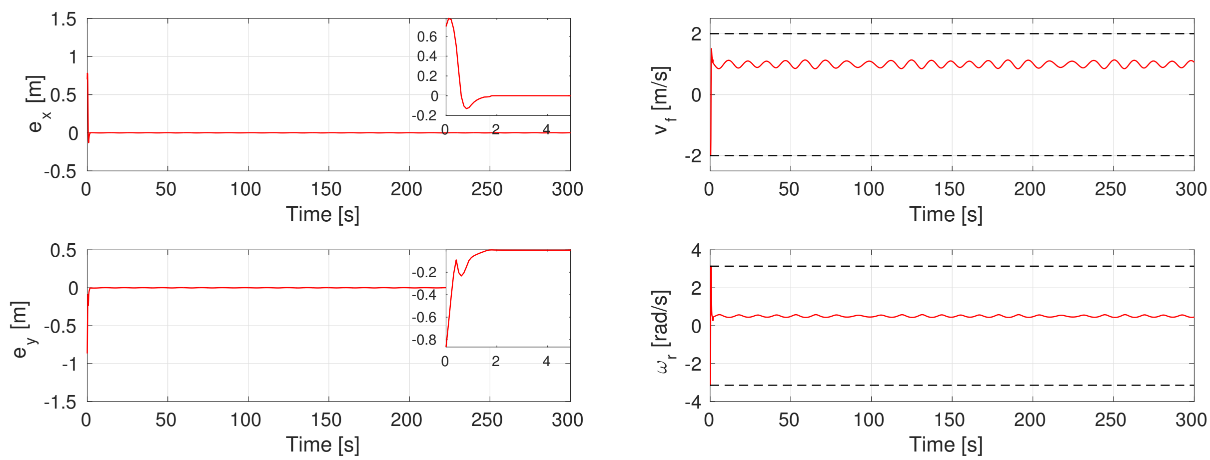
V. Simulation Results

The efficacy of the proposed NMPC law is demonstrated in simulations using Matlab based simulation software VirtualArena [18]. A scenario is considered where the robotic vehicle is required to orbit around a moving target. It is assumed that the target trajectory is known a priori. Such a scenario can be anticipated in situations such as single beacon navigation and tracking problem where the robotic vehicle has to execute maneuvers that makes the target states observable [9], although in such situations the target trajectory is not known a priori. Nevertheless, the presented control method in its state feedback form is an important step towards the development of an output feedback control solution where the target position and velocity are estimated on-line through appropriate selection of the estimation strategy. In order to demonstrate the flexibility of the MPF framework, another scenario is considered wherein the robotic vehicle is tasked to follow a moving lemniscate path. Such scenarios arise in applications such as single/multiple



(a) Circular Path **(b) Lemniscate Path**

Fig. 2 Position of the robotic vehicle, target and the reference moving path for the two scenarios



(a) Error variable e(t) **(b) Computed NMPC control inputs**

Fig. 3 Error variable and the computed control inputs for the circumnavigation scenario

target tracking by a UAV [3]. The position of the target $\mathbf{p}_t^I(t)$ and the desired path $\mathbf{p}_d^T(\gamma)$ used in the simulations for the two scenarios presented above are shown in Table 1. The sampled-data NMPC was simulated with sampling period of 0.1 seconds for simulation time of 300 and 50 seconds for circular and lemniscate scenarios respectively. The horizon length was chosen as 0.3 seconds. The inputs were constrained as $v_{\max} = 2$ [m/s] and $\omega_{\max} = \pi$ [rad/s]. The gain matrix in the auxiliary control law was set to $K_p = 0.1I_{2\times2}$. The weighing matrices Q and R in the cost function was set to $Q = 10I_{2\times2}$ and $R = I_{2\times2}$. The asymptotic tracking error term $\boldsymbol{\epsilon} = [0.2, \quad 0]'$ was used to compute the error variable $\mathbf{e}(t)$. For the given simulation parameters, Figure 2a and 2b shows the position of the reference path and the path followed by the robotic vehicle for circular and lemniscate scenario respectively. Clearly the robot is able to track the moving target while following the desired geometric path. Additionally, the proposed approach is flexible to different path specifications parameterized by a path variable γ. The error variables and computed NMPC control inputs are shown in Figure 3a and 3b respectively. The NMPC control law is able to asymptotically drive the error to zero and hence $\|\mathbf{p}_r^I - \mathbf{p}_d^I(t,\gamma)\| \to \|\boldsymbol{\epsilon}\|$ as explained in Remark 1. Also, the control inputs satisfy the imposed constraints.

Table 1 Target position and desired path

	Circular Path	Lemniscate Path
Target Position $\mathbf{p}_t^I(t)$	$(0.1t, \quad 2\sin(0.05t))'$	$(4, \quad 0.1t)'$
Desired Path $\mathbf{p}_d^T(\gamma)$	$(2\cos(0.5\gamma), \quad 2\sin(0.5\gamma))'$	$\left(\frac{\cos(0.5\gamma)}{1+\sin(0.5\gamma)^2}, \quad \frac{\sin(0.5\gamma)\cos(0.5\gamma)}{1+\sin(0.5\gamma)^2}\right)'$

VI. Conclusion

In this paper, a MPF motion control problem was considered for a constrained under-actuated vehicle. A continuous time sampled-data NMPC control law was designed and its stability properties was outlined through appropriate design of the auxiliary control law, stage cost function, and terminal cost function. Furthermore, it was shown that the terminal constraint set can be excluded by choosing an input constraint set such that the auxiliary control is always feasible. The results were illustrated through simulations with two scenarios: moving circular path and lemniscate path. Although the results were presented for 2D case, the results for 3D case are straightforward. Future work would involve relaxation of the assumption that the target trajectory is known a priori and instead could be estimated, resulting in a output feedback controller. It is also interesting to consider the effect of disturbances such as wind gusts for aerial vehicles, and water currents for the underwater case, on the performance of the proposed controller.

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