U.PORTO
FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Differential Inclusion Approach for Mixed Constrained Problems Revisited

Report FEUP/DEEC/ISR/MRP1/2013

Maria do Rosário de Pinho, Igor Kornienko

Dec. 2013

Abstract

Properties of control systems described by differential inclusions are well established in the literature. Of special relevance to optimal control problems are properties concerning measurability, convexity, compactness of trajectories and Lipschitz continuity of the set-valued mapping (or multifunction) defining the differential inclusion of interest. In this work we concentrate on dynamic control systems coupled with mixed state-control constraints. We characterize a class of such systems that can be described by an appropriate differential inclusion defined by a set-valued mapping exhibiting "good" properties. We illustrate the importance of our findings with respect to existence of solution of optimal control problems.

1 Introduction

Control systems described in terms of differential inclusions have been extensively studied in the literature (see, e.g., [2, 3, 7, 9, 16, 18, 21, 22] to name but a few). Differential inclusions appear in control theory when dynamical systems are expressed as

$$\dot{x}(t) \in F(t, x(t)), \tag{1}$$

where $t \in I \subset \mathbb{R}$, $x \in \mathbb{R}^n$ and F is a set-valued mapping (or multifunction) with closed values in $\mathbb{R}^n$. Such systems make it possible to study in a uniform way a large number of control problems (in this respect see for example [7]). Differential inclusion control problems have proved to be a useful framework for optimal control. For example, they are convenient to state conditions under which existence of solution is ensured and an useful tool to derive optimality conditions (see [7] and [22], for example).

It is commonly accepted that differential inclusions are a ÂťÂťnatural framework" to study dynamical systems with mixed state-control constraints (see [22], pp. 38). Such approach has been used for example in [9], [13] and, recently, in [10], [11] and [12]. A central issue in this respect is the requirement that differential inclusions have certain properties. In particular, it is essential to establish under which conditions the set of trajectories of control systems described in terms of ordinary differential equations coincide with the set of trajectories satisfying (1). In this respect many questions arise as those of measurability of the set-valued mapping defining the differential inclusion (so existence of measurable selections is guaranteed), compactness of trajectories, convexity properties (two subjects relevant for the existence of solution to optimal control problems), etc. Although such aspects are clearly and concisely treated in the literature for control systems of the form

$$\begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [a, b], \\ u(t) &\in U(t) \quad \text{a.e.} \quad t \in [a, b], \end{cases}$$

(see for example Chapter 2 in [22]), the same cannot be said when control systems are coupled with mixed constraints. The system of interest, herein denoted as (Σ), involves the dynamics

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [a, b],$$

mixed constraints

$$(x(t), u(t)) \in S(t) \quad \text{a.e.} \quad t \in [a, b], \tag{2}$$

and boundary conditions

$$(x(a), x(b)) \in E. \tag{3}$$

The data comprises a fixed interval $[a, b]$, a function $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, a set-valued mapping $S : [a, b] \to \mathbb{R}^n \times \mathbb{R}^k$ and a set $E \subset \mathbb{R}^n \times \mathbb{R}^n$. For such system, a pair (x, u) comprising an absolutely continuous function x (the state trajectory) and a measurable function u (the control), will be called throughout a *process* if it satisfies all the constraints of the above system.

Our aim is to establish conditions on the data of (Σ) that translate on useful properties for the set-valued mappings defining the corresponding differential inclusion. To highlight the required properties, while keeping exposition as simple as possible, we work under assumptions that although strong in terms are nevertheless common in applications. In particular, we restrict our analysis to systems generating bounded set-valued mappings. Although, for mixed constrained system, unbounded set-valued mappings may be of use in some cases (and for such cases we refer the reader to [9], [10], [11] and [12]), systems with bounded set-valued mapping remain of interest for applications.

Clearly, conditions under which the state trajectories for (Σ) coincide with the trajectories of a certain differential inclusion

$$\dot{x}(t) \in F_m(t, x(t)) \quad \text{a.e.} \quad t \in [a, b],$$

(where F_m is a set-valued mapping to be defined shortly) satisfying the boundaries constraints (3) will be central in our analysis. We shall pay particular attention to the case where

$$S(t) := \{(x, u) \in \mathbb{R}^n \times U : \; g(t, x(t), u(t)) \leqslant 0\}, \tag{4}$$

where $U \subset \mathbb{R}^k$ and $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$. However, we do not limit our discussion to this case.

We emphasize that the contribution of this paper does not reside only the novelties of the results but rather on gathering them together as well as on the presentation of its proofs. Our work also highlights the importance of a *bounded slope* condition imposed on the mixed constraints in the vein [9] and its relation with Lipschitz properties of the set-valued mappings in our setting. Another aspect of relevance in our discussion resides on convexity assumptions on set-valued mappings, a discussion accompanied by simple but illustrative examples.

This paper is organized in the following way. In the next section we state an auxiliary result that will be relevant in our analysis. In section 3, we introduce the main ingredients of our work as well as the main assumptions. More assumptions, this time for the case when $S(t)$ is as in (4), are presented in section 4 followed by some results relating these assumptions with those in the previous section. Convexity of various set-valued mapping related to our task are discussed in section 6. In section 7 we establish measurability and Lipschitz properties of the relevant set-valued mappings. The compactness of trajectories as well as equivalence of the set of trajectories of (Σ) and that of the corresponding differential inclusion are presented in section 8. We finish this paper with section 8 where we mention some applications of our work and we discuss future work.

Notations: If $g \in \mathbb{R}^m$, the inequality $g \leqslant 0$ is interpreted component-wise. Define $\mathbb{R}^m_- = \{\xi \in \mathbb{R}^m : \xi \leqslant 0\}$ and likewise for $\mathbb{R}^m_+$. The closed ball centred at x with radius δ is denoted $\bar{B}(x, \delta)$ and likewise for the open ball regardless of the dimension of the underlying space. On the other hand, $\bar{B}$ and B denote the closed and open unit ball centred at the origin. Also $|\cdot|$ is the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$. For any closed set $A \subset \mathbb{R}^p$ the distance of a point $x \in \mathbb{R}^p$ to the set A is defined as

$$d_A(x) = \inf\{|x - a| : \ a \in A\}.$$

If $\Omega \subset \mathbb{R}^p$ and $F : \Omega \to \mathbb{R}^q$ is a set-valued mapping, then the graph of F is defined as

$$\text{Gr } F := \{(x, y) \in \Omega \times \mathbb{R}^q : \ y \in F(x)\}.$$

We say that a set $S \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ is $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ -measurable when we refer to measurability relative to the σ-field generated by the products of Lebesgue measurable subsets in $\mathbb{R}$, Borel measurable subsets in $\mathbb{R}^n$ and Borel measurable subsets in $\mathbb{R}^m$.

Consider now a function $h : [a, b] \to \mathbb{R}^p$. We say that $h \in W^{1,1}([a, b]; \mathbb{R}^p)$ if and only if it is absolutely continuous; $h \in L^1([a, b]; \mathbb{R}^p)$ iff h is integrable; and $h \in L^\infty([a, b]; \mathbb{R}^p)$ iff h is essentially bounded. The norm of $L^1([a, b]; \mathbb{R}^p)$ is denoted by $\|\cdot\|_1$ and the norm of $L^\infty([a, b]; \mathbb{R}^p)$ is $\|\cdot\|_\infty$.

Take $A \subset \mathbb{R}^n$ to be a closed set with and consider $x^* \in A$. Also let $f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ be a a lower semicontinuous function. With respect to f, $x^* \in \mathbb{R}^k$ will be such that $f(x^*) < +\infty$. Concerning nonsmooth analysis we use the following notation: $N^P_A(x^*)$ is the *proximal normal cone to* A at x^*, $N^L_A(x^*)$ is the *limiting normal cone* to A at x^*, $N^C_A(x^*)$ is the *Clarke normal cone to* A at x^*, $\partial^L f(x^*)$ is *limiting subdifferential* of f at x^* and $\partial^C f(x^*)$ is *(Clarke) subdifferential* of f at x^*. If f is Lipschitz continuous near x^*, the convex hull of the limiting subdifferential, co $\partial^L f(x^*) = \partial^C f(x)$.

2 Auxiliary Result

Before proceeding we state an adaptation of Theorem 3.5.2 in [9] that will be important in the forthcoming analysis.

Consider a set-valued mapping $\Gamma : [a, b] \times \mathbb{R}^n \to \mathbb{R}^k$. For each $t \in [a, b]$, consider the set-valued mapping $x \to \Gamma(t, x)$ and suppose that the graph $G(t)$ of this set-valued mapping is closed. Suppose that $u^* \in \Gamma(t, x^*)$ and that the following condition holds:

(BS') There exist $\varepsilon > 0$, $R > 0$ and $\mathcal{K} > 0$ such that, for almost every t,

$$x \in B(x^*, \varepsilon), \quad u \in B(u^*, R), \quad (\alpha, \beta) \in N^P_{G(t)}(x, u) \implies |\alpha| \leqslant \mathcal{K}|\beta|.$$

We emphasize that we are assuming the parameters R and $\mathcal{K}$ to be independent of t. As the reader may suspect a more general definition involves such parameters as (measurable) functions of t. Here and throughout this paper, we shall remove the dependency of t of many parameters in our assumptions as long as this allows us to skip some technical details while retaining their significance. For a complete discussion on bounded slope condition [BS'] and pseudo- Lipschitz properties of set-valued mappings we refer the reader to [9] and [12].

The following theorem asserts that such a set-valued mapping $x \to \Gamma(t, x)$, satisfying (BS'), is pseudo-Lipschitz.

Theorem 2.1 (adaptation of Theorem 3.5.2 in [9]) *Let $x \to \Gamma(t,x)$ satisfy (BS') near $(x^*,u^*) \in G(t)$. Then for any $\xi \in]0,1[$ and any $x_1, x_2 \in B(x^*, \bar{\varepsilon})$ the following holds*

$$\Gamma(t,x_1) \cap \bar{B}(u^*,(1-\xi)R) \subset \Gamma(t,x_2) + \mathcal{K}|x_1 - x_2|\bar{B},$$

where $\bar{\varepsilon} = \min\{\varepsilon, \xi R/3\mathcal{K}\}$.

3 Main Assumptions

Mixed constraints, also known as state dependent control constraints, can be written in the general form as (2). We associate with S the set-valued mapping $S_m : [a,b] \times \mathbb{R}^n \to \mathbb{R}^k$ defined as

$$S_m(t,x) = \left\{u \in \mathbb{R}^k : \ (x,u) \in S(t)\right\}.$$

For each $t \in [a,b]$ the set $S(t)$ is the graph of $x \to S_m(t,x)$, that is,

$$(x,u) \in S(t) \iff u \in S_m(t,x).$$

Additionally, introduce the set-valued mapping

$$F_m(t,x) : \ = \ \{f(t,x,u) : \ u \in S_m(t,x)\} \tag{5}$$

The set-valued mapping (8) will be of importance when we concentrate on $S(t)$ as in (4). Take any absolutely continuous function $x^* : [a,b] \to \mathbb{R}^n$ and define

$$X(t) := x^*(t) + \varepsilon\bar{B} \quad \text{and} \quad S^*_\varepsilon(t) := S(t) \cap \big((x^*(t) + \varepsilon B) \times \mathbb{R}^k\big). \tag{6}$$

It is important to notice that $S^*_\varepsilon(t)$ is defined as the intersection of $S(t)$ with the open ball $x^*(t) + \varepsilon B$, not with $X(t)$.

When appropriate, we shall impose that the function x^* satisfy the differential inclusion

$$\dot{x}^*(t) \in F_m(t, x^*(t)). \tag{7}$$

We now state several assumptions that will be use in the forthcoming analysis. Let $\phi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^p$ be a general function (ϕ may then replaced by f or g).

(B1) The function $t \to \phi(t,x,u)$ is $\mathcal{L}$-measurable for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^k$.

(B2) The set-valued mapping S is $\mathcal{L}$-measurable and, for each $t \in [a,b]$, $S(t)$ is closed.

(B3) The set E is closed.

(B4) For each $t \in [a,b]$ and $x \in X(t)$, there exists $u \in \mathbb{R}^k$ such that $(x,u) \in S(t)$. Furthermore, each $t \in [a,b]$ there exists a constant σ such that

$$(x,u) \in S(t) \implies |u| < \sigma.$$

(BS) There exists a constant $\mathcal{K} > 0$ such that, for almost every $t \in [a,b]$ and all $(x,u) \in S^*_\varepsilon(t)$,

$$(\alpha, \beta) \in N^P_{S(t)}(x,u) \implies |\alpha| \leqslant \mathcal{K}|\beta|.$$

(CA) For all $t \in [a,b]$ and $x \in X(t)$, $F_m(t,x)$ is convex.

(LC) There exist constants k^ϕ_x and k^ϕ_u such that, for almost every $t \in [a,b]$ and all $(x_i,u_i) \in S^*_\varepsilon(t)$ $(i = 1,2)$, we have

$$|\phi(t,x_1,u_1) - \phi(t,x_2,u_2)| \leqslant k^\phi_x|x_1 - x_2| + k^\phi_u|u_1 - u_2|.$$

Some of the above assumptions could (in some situations) be stated in weaker forms. The strengthening of the assumptions is done in the second part of (B4) (where an uniform bound on the u is added) and in the second part with respect to the nature of the constant in (BS) and (LC). Indeed, the parameters $\mathcal{K}$ in (BS) and k^ϕ_x and k^ϕ_u in (B1) are constants while in [10] such parameters are considered to be merely measurable functions. As mentioned before, the strengthening of the hypotheses is introduced to allow us to omit technical details in the forthcoming analysis.

4 On $S(t)$ defined by (4)

We now concentrate on the case where the set $S(t)$ is given by (4). Throughout we impose that the function g defining the set $S(t)$ satisfies (B1) and (LC). This will be stated when needed.

We start our analyses introducing the set $U \subset \mathbb{R}^k$ and the set-valued mapping $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ and $F_m : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ defined as

$$F(t,x) : \quad = \quad \{(f(t,x,u),\ g(t,x,u)) :\ u \in U\}, \tag{8}$$

and additional assumptions on U and F:

(ICU) The set U is compact and, for each $x \in \mathbb{R}^n$, there exists a $u \in U$ such that $g(t,x,u) \leqslant 0$.

(ICC) For all $t \in [a,b]$ and $x \in X(t)$, $F(t,x)$ is convex.

We shall also use the assumption

(IMC) There exists a constant M such that, for almost every t, all $(x,u) \in S^*_\varepsilon(t)$, $\eta \in N^L_U(u)$, $\gamma \in \mathbb{R}^m_+$ with $\langle \gamma,\ g(t,x,u)\rangle = 0$, we have

$$(\alpha, \beta - \eta) \in \partial^L_{x,u} \langle \gamma,\ g(t,x,u)\rangle \implies |\gamma| \leqslant M|\beta|.$$

We refer the reader to [10] for restatements and discussion of (IMC) when $(x,u) \to g(t,x,u)$ is continuously differentiable. There, (IMC) is related to well known linear independence condition of the gradients of g.

Next we shall relate these assumptions with previous ones imposed on $S(t)$ and F_m.

Lemma 4.1 *Consider $S(t)$ as defined by (4). Assume that g satisfies (B1) and (LC) and that (ICU) holds. Then $t \to S(t)$ is a Lebesgue measurable set-valued mapping and, for each t, the set $S(t)$ is nonempty and closed.*

Remark: For S as defined by (4) Lemma 4.1 states conditions on g and U implying that (B2) and (B4) hold.

Proof. For each $t \in [a,b]$, $S(t)$ is nonempty by (ICU). By (LC) we know that g is a Carathéodory function. Then Proposition 14.33 in [20] asserts that $S(t)$ is a closed set for each t and $t \to S(t)$ is Lebesgue measurable. $\square$.

Our task now is to investigate the relation between (IMC) and (BS). The following characterization of $N^L_{S(t)}(x,u)$ will be a cornerstone in this respect.

Lemma 4.2 *Consider $S(t)$ as defined by (4). Assume (ICU) and (IMC) hold and that g satisfies (B1) and (LC). Then for almost every $t \in (a,\ b)$, for all $(x,u) \in S^*_\varepsilon(t)$ and all $(\alpha,\beta) \in N^L_{S^*_\varepsilon(t)}(x,u)$, there exists an $\gamma \geqslant 0$ with $\langle \gamma, g(t,x,u)\rangle = 0$ such that*

$$(\alpha,\beta) \in \partial^L_{(x,u)} \langle \gamma, g(t,x,u)\rangle + \{0\} \times N^L_U(u). \tag{9}$$

We postpone the proof of this Lemma (and it will appear in the end of this section) and we go straight to an important consequence of it relating (IMC) with (BS).

Corollary 4.3 *Under the assumptions of Lemma 4.2, (BS) holds.*

Proof. Take $t \in [a,b]$ where the properties of (IMC) and (LS) hold. Take also any $(\alpha,\beta) \in N^P_{S(t)}(x,u)$. Since $S^*_\varepsilon(t) \subset S(t)$ we have $N^P_{S(t)}(x,u) \subset N^P_{S^*_\varepsilon(t)}(x,u)$. On the other hand, we also have $N^P_{S^*_\varepsilon(t)}(x,u) \subset N^L_{S^*_\varepsilon(t)}(x,u)$ (by Proposition 4.2.6 in [22]). Thus $(\alpha,\beta) \in N^L_{S^*_\varepsilon(t)}(x,u)$ and it follows from Lemma 4.2 and (IMC) that for $\gamma \geqslant 0$ with $\langle \gamma, g(t,x,u)\rangle = 0$,

$$\eta \in N^L_U(u), \quad (\alpha, \beta - \eta) \in \partial^L_{(x,u)} \langle \gamma, g(t,x,u)\rangle \Longrightarrow |\gamma| \leqslant M\ |\beta|.$$

By (LC) $(x,u) \to \langle \gamma, g(t,x,u)\rangle$ is Lipschitz continuous with constant $|\gamma| \max\{k^g_x, k^g_u\}$. Appealing to the properties of subdifferentials, we deduce that

$$|\alpha| \leqslant |(\alpha, \beta - \eta)| \leqslant \max\{k^g_x, k^g_u\}|\gamma| \leqslant \max\{k^g_x, k^g_u\} M |\beta|,$$

proving that (BS) holds with $\mathcal{K} = \max\{k_x^g, k_u^g\}M$. $\square$

Proof of Lemma 4.2: Let $t \in [a,b]$ be such that the property in (IMC) holds. Let $\varphi(x,u) = g(t,x,u)$ and set

$$C_1(t) = \varphi^{-1}(\mathbb{R}^m_-) \quad \text{and} \quad C_2(t) = X(t) \times U.$$

Now take any

$$(x,u) \in S(t), \quad x \in x^*(t) + \varepsilon B \quad \text{and} \quad (\alpha, \beta) \in N^L_{S^*_\varepsilon(t)}(x,u).$$

Observe that $x \in X(t)$ but it is not on the boundary of X (we are invoking (IMC)). This will be of importance in what follow.

Our next task is to characterize $N^L_{C_1(t)}(\varphi(x,u))$ in terms of $\partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle$. Corollary 10.50 in [20] is essential here. To apply such result, we first claim that if $\gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u))$ such that $(0,0) \in \partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle$, then $\gamma = 0$. To see this take any γ such that $(0,0) \in \partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle$. Since $\gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u))$, we have

$$\langle \gamma, \varphi(x,u)\rangle = 0, \quad \gamma \geqslant 0.$$

By (IMC) we deduce from $(0,0) \in \partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle$ and the fact that $0 \in N^L_U(u)$, that $|\gamma| \leqslant 0$. It follows that $\gamma = 0$ and the conditions under which Corollary 10.50 in [20] holds are then satisfied. Applying it we then conclude that

$$N^L_{C_1(t)}(x,u) \subset \bigcup \left\{ \partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle : \ \gamma \in N^L_D(\varphi(x,u)) \right\}, \tag{10}$$

meaning that if $(v_1, v_2) \in N^L_{C_1(t)}(x,u)$, then there exists a $\gamma \geqslant 0$ such that $\langle \gamma, \varphi(x,u)\rangle = 0$ and $(v_1, v_2) \in \partial^L\langle \gamma, \varphi(x,u)\rangle$.

Our next task is to prove that $N^L_{C_1(t)}(x,u)$ and $N^L_{C_2(t)}(x,u)$ are transversal in (x,u), i.e.,

$$(\xi, \zeta) \in -N^L_{C_1(t)}(x,u) \cap N^L_{C_2(t)}(x,u) \implies (\xi, \zeta) = (0,0). \tag{11}$$

Since $N^L_{C_2(t)}(x,u) = N^L_{X(t)}(x,u) \times N^L_U(x,u)$ and $x \in \text{int}X(t)$, we have $\zeta \in N^L_U(x,u)$ and $\xi = 0$. Furthermore, by (10) we have, for some γ with the required properties,

$$(0, -\zeta) \in \partial^L\langle \gamma, \varphi(x,u)\rangle.$$

Invoking (IMC) with $\alpha = 0$, $\beta = 0$ and $\eta = \zeta$, we deduce that $\gamma = 0$. But then $(0, \zeta) = (0,0)$, proving (11). We are now in position to invoke Theorem 6.42 in [20] to conclude that

$$N_{S^*_\varepsilon(t)}(x,u) \subset N^L_{C_1(t)}(x,u) + N^L_{C_2(t)}(x,u).$$

It follows from the above that (9) holds, proving the Lemma. $\square$

5 Convexity Assumption on $F_m(t,x)$

Here we investigate how the convexity of $F_m(t,x)$ (see (CA)) relates with convexity of associated set-valued mappings.

The next two examples show that convexity of $F_m(t,x)$ does not imply convexity of $S_m(t,x)$ nor does the converse hold.

Example 5.1 *Let* $S_m(t,x) = \{u \in (-2,2) : \ -(u^2-1) \leqslant 0\} = [-2,-1] \cup [1,2]$ *and define* F_m *as* $F_m(t,x) := \left\{u^2 : u \in S_m(t,x)\right\}$. *Then* $F_m(t,x) = [1,4]$ *is convex although* $S_m(t,x)$ *is not.*

Example 5.2 *Now take* $S_m(t,x) = [-1,0]$. *This set is convex but* $F_m(t,x) = \{(u,u^2) : \ u \in S_m(t,x)\}$ *is not;* $F_m(t,x)$ *coincides with the graph of* $m(u) = u^2$.

It is worth mentioning that convexity of F_m is a property of the geometry of the set and it does not imply the convexity of the function $u \to f(t,x,u)$. In example 5.2, $f(t,x,u) = (u,u^2)$ and the two functions $f_1(t,x,u) = u$ and $f_2(t,x,u) = u^2$ are convex. Yet $F_m(t,x)$ fails to be a convex set.

However, with $S(t)$ as defined by (4), convexity of the set U and that of the function $u \to g(t,x,u)$ implies convexity of $S_m(t,x)$ (a simple matter to prove). But again, this alone is not enough to guarantee that (CA) holds and so we need to impose it when needed.

Although for a general $S(t)$, (CA) may be difficult to check, there are special cases where (CA) is implies by conditions easier to verify. This happens when $S(t)$ is as defined by (4). In that case, (IMC) may be easier to verify and (IMC) implies (CA). However the converse does not hold as we show in Lemma 5.3 below. Given this structure of $S(t)$ one may be tempted to think that other easier verifiable convexity conditions would involve the set-valued mappings

$$F^f(t,y) = \{f(t,y,u) : \ u \in U\}, \quad G^g(t,y) = \{g(t,y,u) : \ u \in U\}. \tag{12}$$

Next we investigate how convexity properties of F, F_m, F^f and G^g, in the particular case where $S(t)$ is defined by (4), relate to each other next.

Lemma 5.3 *Consider any $t \in [a,b]$ and $x \in X(t)$ such that $S_m(t,x) \neq \emptyset$ where $S(t)$ is defined as*

$$S(t) := \{(x,u) \in \mathbb{R}^n \times U : \ g(t,x(t),u(t)) \leqslant 0\}.$$

Let F^f and G^g be as in (12). The following relations hold:

1. *$F(t,x)$ convex $\Longrightarrow F_m(t,x)$ convex, but the opposite implication does not hold.*
2. *$F(t,x)$ convex $\Longrightarrow F^f(t,x)$ and $G^g(t,x)$ are convex, but the opposite implication does not hold.*
3. *The convexity of $F_m(t,x)$ does not imply the convexity of $F^f(t,x)$ and $G^g(t,x)$ and the opposite implication does not hold.*

Proof.

1. $F(t,x)$ **convex** $\Longrightarrow F_m(t,x)$ **convex.**

   Take any $v_1,\ v_2 \in F_m(t,x)$. Then there exist $u_1,\ u_2 \in U$ such that $v_1 = f(t,x,u_1)$, $v_2 = f(t,x,u_2)$, $g(t,x,u_1) \leqslant 0$ and $g(t,x,u_2) \leqslant 0$. Set $z_i = g(t,x,u_i)$, $i = 1,2$. We have $(v_i,z_i) \in F(t,x)$, $i = 1,2$. Since $F(t,x)$ is convex, for any $\beta \in (0,1)$, there exists $u \in U$ such that $(v,z) = \beta(v_1,z_1) + (1-\beta)(v_2,z_2) = (f(t,x,u), g(t,x,u))$. But $z = \beta z_1 + (1-\beta)z_2 = g(t,x,u) \leqslant 0$. Thus $v \in F_m(t,x)$ proving convexity of $F_m(t,x)$.

   $F_m(t,x)$ **convex** $\not\Longrightarrow F(t,x)$ **convex.**

   Take $U = [-1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = -(u+1)^2$. Then for any x, we have $S_m(t,x) = [-1,1]$ and $F_m(t,x) = [-1,1]$, both convex sets. However,

   $$F(t,x) = \left\{(u, -(u+1)^2) : \ u \in U\right\}$$

   is not a convex set.

2. $F(t,y)$ **convex** $\Longrightarrow F^f(t,y)$ **and** $G^g(t,y)$ **are convex.**

   Fix y and take any $v_1,\ v_2 \in F(t,x)$. Then there exist $u_1,\ u_2 \in U$ such that $v_1 = f(t,x,u_1)$ and $v_2 = f(t,x,u_2)$. Set $z_1 = g((t,x,u_1)$ and $z_2 = g(t,x,u_2)$. Then, for any $\beta \in [0,1]$ $(v,z) = \beta(v_1,z_1) + (1-\beta)(v_2,z_2)$ is such that $(v,z) \in F(t,x)$, i.e, there exists $u \in U$ such that $(v,z) = (f(t,x,u), g(t,x,u))$. It follows that $v \in F^f(t,x)$ and $z \in G^g(t,x)$ proving convexity of $F^f(t,x)$ and $G^g(t,x)$.

   $F^f(t,x)$ **and** $G^g(t,x)$ **convex** $\not\Longrightarrow F(t,x)$ **convex.**

   To see this it is enough to define $U = [-1,1]$, $f(t,x,u) = u^2$, and $g(t,x,u) = u$. Then $F^f(t,x) = [0,1]$, $G^g(t,x) = [-1,1]$ and $F(t,x) = \{(u^2,u) : \ u \in [-1,1]\}$, not convex.

3. $F_m(t,x)$ **convex** $\not\Longrightarrow F^f(t,x)$ **and** $G^g(t,x)$ **convex.**

   Take $U = [-1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = (-u, u^3 - u)$. Then both $S_m(t,x) = [0,1]$ and $F_m(t,x) = [0,1]$ are convex. On the other hand, although $F^f(t,x) = [0,1]$ is convex, we do not have convexity of $G^g(t,x) = \{(-u, u^3-u) : \ u \in [-1,1]\}$.

   $F^f(t,x)$ **and** $G^g(t,x)$ **are convex** $\not\Longrightarrow F_m(t,x)$ **convex.**

   Take $U = (-1,1)$, $f(t,x,u) = u$ and $g(t,x,u) = -u^2 + 1/4$. Then $F^f(t,x) = [-1,1]$ and $G^g(t,x) = [-3/4, 1/4]$ are both convex. However, $F_m(t,x) = [-1,-1/2] \cup [1/2,1]$ is not convex.

We summarize our findings:

$$
\begin{array}{|cccccc|}
\hline
F(t,x) \quad \text{convex} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & F^f(t,x),\ G^g(t,x) \text{ convex} & & & \\
\Downarrow \ \not\Uparrow & & \not\Uparrow \ \not\Downarrow & & & \\
F_m(t,x) \quad \text{convex} & \Longleftrightarrow & F_m(t,x) \text{ convex} & & \begin{array}{c}\not\Longrightarrow \\ \not\Longleftarrow\end{array} & S_m(t,x) \text{ convex} \\
\hline
\end{array}
$$

6 Properties of the set-valued Mappings

We establish important properties of the set-valued mappings S and F_m. We focus on a general S. Taking into account the results of the previous section, our results also applied when $S(t)$ is defined by (4).

Lemma 6.1 *Assume that (B2) and (B4) hold and that f satisfies (B1) and (LC). Then*

1. *For almost every $t \in [a,b]$ and each $x \in X(t)$, the sets $S_m(t,x)$ and $F_m(t,x)$ are nonempty and compact.*
2. *The set-valued mapping F_m is $\mathcal{L} \times \mathcal{B}$-measurable.*
3. *The graph of $(t,x) \to S_m(t,x)$ is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$-measurable set.*
4. *Assume that x^* satisfies (7). Then, for almost every $t \in [a,b]$ and all $x(t) \in X(t)$, there exists an integrable function c such that for all $\gamma(t) \in F_m(t,x(t))$ we have $|\gamma(t)| \leqslant c(t)$.*

Proof. Non emptiness and compactness of $S_m(t,x)$ follows from (B2) and (B4). Also, (B4) guarantees that $F_m(t,x)$ is non empty. Taking into account that $u \to f(t,x,u)$ is continuous by (LC), we get the compactness of the set $F_m(t,x)$.

We now turn to 2 of the Lemma. Take any compact set $A \subset \mathbb{R}^n$. We want to prove that

$$\{(t,x) \in [a,b] \times \mathbb{R}^n : \ F_m(t,x) \cap A \neq \emptyset\} \tag{13}$$

is $\mathcal{L} \times \mathcal{B}$-measurable. Assume that $f^{-1}(A) \neq \emptyset$. By (B1) and (LC), $t \to f(t,x,u)$ is measurable for each (x,u) and $(x,u) \to f(t,x,u)$ is continuous for almost every t. Thus Proposition 2.3.6 in [22] asserts that f is an $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$-measurable function. It follows that

$$f^{-1}(A) = \left\{(t,x,u) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^k : \ f(t,x,u) \in A\right\}$$

is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurable set. On the other hand, the set-valued mapping $t \to S(t)$ is $\mathcal{L}$-measurable and closed valued by (B2). By Theorem 2.3.7 in [22], the graph of $t \to S(t)$,

$$\Upsilon := \{(t,x,u) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^k : \ (x,u) \in S(t)\}, \tag{14}$$

is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurable set and, consequently, $f^{-1}(A) \cap \Upsilon$ is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$-measurable set. Now, take any $t,x) \in [a,b] \times \mathbb{R}^n$ such that $F_m(t,x) \cap A \neq \emptyset$. Then (B4) guarantees the existence of a $u \in \mathbb{R}^k$ such that $(x,u) \in S(t)$. It follows that $(t,x,u) \in f^{-1}(A) \cap \Upsilon$. Since (t,x) belongs to (13) if and only if there is a $u \in \mathbb{R}^k$ such that $(t,x,u) \in f^{-1}(A) \cap \Upsilon$ we deduce from Proposition 2.34 in [14] the $\mathcal{L} \times \mathcal{B}$ measurability of F_m.

Statement 3 of the Lemma follows from the $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurability of the set (14) and the fact that $(x,u) \in S(t)$ is equivalent to $u \in S_m(t,x)$.

Now it remains to prove 4. Take $t \in [a,b]$ such that $\dot{x}^*(t) \in F_m(t, x^*(t))$ (see (7)). Let u^* be such that $u^* \in S_m(t,x^*(t))$ and $\dot{x}^*(t) = f(t,x^*(t),u^*)$. Take x such that $x \in X(t)$. Since by (B4) we have $F_m(t,x) \neq \emptyset$, take any $\gamma \in F_m(t,x)$. By definition of F_m there there exists a $u \in S_m(t,x)$ such that $\gamma = f(t,x,u)$. Appealing to (LC) we now have

$$|\gamma| \leqslant |f(t,x^*(t),u^*)| + 2k_x^f \varepsilon + 2k_u^f \sigma = |\dot{x}^*(t)| 2k_x^f \varepsilon + 2k_u^f \sigma.$$

Set $c(t) = |\dot{x}^*(t)| + 2k_x^f \varepsilon + 2k_u^f \sigma$. Observe that upper bound does not depend on the choice of x or u and it holds for almost every t. Since $\dot{x}^*$ is an integrable function we conclude that $c \in L^1$ proving our claim. $\square$

Remark: In 4 of Lemma 6.1 the requirement that x^* satisfies (7) is added to guarantee the integrability of c. Alternatively, we can stipulate that f is uniformly bounded by an integrable function b. $\square$

We now investigate Lipschitz properties of $x \to S_m(t,x)$ and $x \to F_m(t,x)$ for each t. In this respect, (BS) is essential as we shall see. Indeed, conditions (B1), (B2), (B4) and (LC) by themselves, are not enough to guarantee lower semi-continuity of $x \to S_m(t,x)$ or $x \to F_m(t,x)$, let alone Lipschitz continuity, as the following example shows.

Example 6.2 *Let us fix $t \in [a,b]$ (the interval $[a,b]$ here has no relevance) and set $S(t) = \{(x,u) \in \mathbb{R}^n \times \mathbb{R} : \ u \in [-1,1], \ u|x| \leqslant$ Since, for each t,*

$$S_m(t,x) = \begin{cases} [-1,1] & \text{if} \quad x = 0, \\ [-1,0] & \text{if} \quad x \neq 0, \end{cases}$$

we have $F_m(t,x) = \{x+u : \ u \in S_m(t,x)\}$. It is a simple matter to see that (B1), (B2), (B4) hold and that $f(x,u) = x+u$ satisfies (LC). However, both F_m and S_m fail to be lower semi-continuous. To see that, consider any sequence $\{x_i\}$ such that $x_i \neq 0$ and $x_i \to 0$. Then $1/2 \in S_m(t,0)$ and $1/2 \in F_m(t,0)$. But there is no convergent sequence $\{u_i\}$ with limit equal to $1/2$, since $u_i \leqslant 0$. Consequently, there is no sequence $\gamma_i \in F_m(t,x_i)$ converging $1/2$.

Assumption (BS) excludes this example from our context. Indeed, for any t, we have $(1,0) \in N^P_{S(t)}(0,1/2)$ and so $1 > 0$ meaning that (BS) is not satisfied. $\square$.

Remark: It is worth mentioning, for future reference, that the set $S(t)$ in the above example can be defined as in (4) where $U = [-1,1]$ and $g(x,u) = u|x|$.

An appeal to Theorem 2.1 asserts that, in our setting, (BS) guarantees that $x \to S_m(t,x)$ is not merely pseudo-Lipschitz: it is in fact Lipschitz continuous as we show next. For completeness we also state a known result: that under our conditions, Lipschitz continuity of $x \to S_m(t,x)$ implies (BS).

Lemma 6.3 *Assume that (B2), (B4) and* **(BS)** *hold. Then there exists constant ε such that, for almost every t,*

$$x, x' \in x^*(t) + \varepsilon B \implies S_m(t,x) \subset S_m(t,x') + \mathcal{K}|x - x'|\bar{B}, \tag{15}$$

where $\mathcal{K}$ is as defined in (BS).

If (B2) and (B4) hold and there exist constant ε and $\mathcal{K}$ such that, for almost every t, (15) is satisfied, then **(BS)** *hold with constant.*

Remark: Before engaging in the proof of this result, it is important to emphasize that the above Lemma is no more than an adaptation of more general results presented and discussed in [9]. We also refer the reader to [12] in this regard.

Proof. Recall that $S(t)$ is the graph of $x \to S_m(t,x)$ and, by (B2), it is a closed set. Now take t such that the property in (BS) holds. Take u^* to be such that $u^* \in S_m(t, x^*(t))$. Such u^* exists by (B4). By (B4) we know that for any $x \in X(t)$ and any $u \in S_m(t,x)$ we have $|u| \leqslant \sigma$. So for any x and u such that $x \in B(x^*(t), \varepsilon)$ (the same ε defining the closed set $X(t)$) and $(x,u) \in S(t)$ we have $u \in B(u^*(t), R)$ with $R = 2\sigma$. It follows from (BS) that (BS') holds with constant $\mathcal{K}$. We can then apply Theorem 2.1, where $\Gamma(t,x) = S_m(t,x)$ and $G(t) = S(t)$, with $\xi = 1/2$. Take $\varepsilon = \min\{\varepsilon, \frac{\sigma}{3\mathcal{K}}\}$ and any $x \in B(x^*(t), \varepsilon)$. Then we have

$$S_m(t,x) \cap \bar{B}(u^*(t), (1-\xi)R) = S_m(t,x) \cap \bar{B}(u^*(t), \sigma) = S_m(t,x)$$

and our first result follows from Theorem 2.1.

We omit the proof of the second part of the Lemma since this can be found in [12]. $\square$

As an immediate conclusion of the above Lemma we get the following Corollary.

Corollary 6.4 *Assume that (B2), (B4) and* **(BS)** *hold and that f satisfies (B1) and (LC). Then there exist ε and k_{F_m} such that, for almost every t,*

$$x, x' \in x^*(t) + B(0,\varepsilon) \implies F_m(t,x) \subset F_m(t,x') + k_{F_m}|x - x'|\bar{B}.$$

Proof. Let $t \in [a,b]$ such that the properties in (BS) and (LC) hold. Let ε as defined in Lemma 6.3 and take any $x, x' \in x^*(t) + \varepsilon B$ and any $\gamma \in F_m(t,x)$ and $\gamma' \in F_m(t,x')$. Let u and u' be such that $(x,u) \in S_m(t,x)$, $(x',u') \in S_m(t,x')$, $\gamma = f(t,x,u)$ and $\gamma' = f(t,x',u')$. By Lemma 6.3 we get

$$|u - u'| \leqslant \mathcal{K}|x - x'|.$$

Then, it follows from Lemma 6.3 that

$$\begin{aligned} |f(t,x,u) - f(t,x',u')| &\leqslant k_x^f|x - x'| + k_u^f|u - u'| \\ &\leqslant k_x^f|x - x'| + k_u^f\mathcal{K}|x - x'| \\ &= (k_x^f + k_u^f\mathcal{K})|x - x'| \end{aligned}$$

and our result follows with $k_{F_m} = k_x^f + k_u^f\mathcal{K}$. □

We now concentrate on measurable selections. We dwell on S_m, although our results can apply to F_m. The following result asserting the existence of measurable and Lipschitz selection of S_m is a direct consequence of Proposition 2.3.10 of [22] and Theorem 9.5.3 in [3].

Lemma 6.5 *Assume that (B2), (B4) and* **(BS)** *hold. Take any measurable function $x : [a,b] \to \mathbb{R}^n$ such that $x(t) \in X(t)$ for each t. Then there exists a measurable function $\mu : [a,b] \to \mathbb{R}^k$ such that*

$$\mu(t) \in S_m(t, x(t)) \quad a.e.\ t \in [a,b]. \tag{16}$$

Furthermore, if for each $t \in [a,b]$ and each $x \in X(t)$, $S_m(t,x)$ is convex, then there exists a constant c and a function $u : [a,b] \times \mathbb{R}^n \to \mathbb{R}^k$ such that $t \to u(t,x)$ is measurable, $x \to u(t,x)$ is Lipschitz with constant $c\mathcal{K}$ and

$$\mu(t) = u(t, x(t)) \quad a.e.\ t \in [a,b].$$

Before presenting the proof it is worth noticing that the convexity of $S_m(t,x)$ is essential here.

Proof. Let Υ denote the graph of $(t,x) \to S_m(t,x)$ and recall that 3 of Lemma 6.1 guarantees that it is a $\mathcal{L}\times\mathcal{B}\times\mathcal{B}$ set. It follows from Proposition 2.34 in [14] that the graph of $t \to S_m(t,x)$ is $\mathcal{L}\times\mathcal{B}$-measurable set. On the other hand, for any $x \in \mathbb{R}^n$, the set $S_m(t,x)$ is nonempty and compact by 1 of Lemma 6.1. Theorem 2.3.7 in [22] then asserts that $t \to S_m(t,x)$ is a measurable set-valued mapping. On the other hand, for each $t \in [a,b]$, Lemma 6.3 guarantees that $x \to S_m(t,x)$ is Lipschitz. We can then apply Proposition 2.3.10 in [22] to deduce the measurability of the set-valued mapping $G(t) = S_m(t,x(t))$. Since $G(t)$ is closed for each t, Theorem 2.3.11 in [22] guarantees the existence of a measurable selection μ of $G(t)$, i.e., μ is a measurable function satisfies (16).

Additionally, suppose that now that $S_m(t,x)$ is convex. Then the last conclusion of follows from Theorem 9.5.3 in [3]. □

7 Differential Inclusions with Mixed Constraints

We are now in position to invoke Chapter 2 in [22] to obtain several relevant properties for our set-valued mapping F_m.

Consider the set $X(t)$ to be defined by x^* satisfying (7)[1]. Recall that the set-valued mapping $X : [a,b] \to \mathbb{R}^n$ is closed and bounded. Under the conditions of Lemma 6.1 it is a simple matter to deduce the compactness of trajectories of F_m as a direct consequence of Theorem 2.5.3 in [22]. For the sake of completeness we state our findings next.

Theorem 7.1 *Assume that (CA) and the conditions under which 4 of Lemma 6.1 hold. Take any sequence $\{x_i\}$, $x_i \in W^{1,1}([a,b];\mathbb{R}^n)$ such that*

$$Gr\ x_i \subset\ Gr\ X, \quad \dot{x}_i(t) \in F_m(t, x_i(t))\ a.a.\ t \in [a,b], \quad x_i(0) \in X(0).$$

Then there exists a subsequence (we do not relabel) such that

$$x_i \to x \text{ uniformly} \quad \text{and} \quad \dot{x}_i \to \dot{x} \text{ weakly in } L^1$$

for some $x \in W^{1,1}([a,b];\mathbb{R}^n)$ such that $\dot{x}(t) \in F_m(t,x(t))$ a.a. $t \in [a,b]$.

[1] Here we need 4 of Lemma 6.1. As remarked before, instead of choosing x^* as a trajectory of F_m, we may consider any absolutely continuous function x^* in the definition of X and add the assumption that there exists an integrable function c such that, for all (x,u) we have $|f(t,x,u)| \leqslant c(t)$ for almost every t.

Our next step if to ensure equivalence between the set of feasible trajectories of system (Σ) and the set of feasible trajectories of F_m. We say that an absolutely continuous function x is a feasible trajectory of F_m if $x(t) \in X(t)$ for all $t \in [a,b]$ and $\dot{x}(t) \in F_m(t, x(t))$ for almost every $t \in [a,b]$. We denote the set of all F_m-feasible trajectories associated with E to be

$$\mathcal{R}^*_{[a,b]}(E) := \{x \in C([a,b];\mathbb{R}^n) : \ x \text{ is an } F_m \text{ trajectory and } (x(a), x(b)) \in E\}.$$

Define $\mathcal{S}^*_{[a,b]}(E)$ to be the set of all absolutely continuous functions x associated with a control $u : [a,b] \to U$ such that $x(t) \in X(t)$ for all $t \in [a,b]$ and (x,u) solves (Σ), i.e,

$$\begin{cases} \dot{x}(t) &=& f(t, x(t), u(t)) \text{ a.e.}, \\ (x(t), u(t)) &\in& S(t) \text{ a.e.}, \\ (x(a), x(b)) &\in& E. \end{cases}$$

Theorem 7.2 *Assume that f satisfies (B1) and (LC) and that (B2)–(B4) and (BS) hold. Then $x \in \mathcal{S}^*_{[a,b]}(E)$ if and only if $x \in \mathcal{R}^*_{[a,b]}(E)$.*

Proof. The implication $x \in \mathcal{S}^*_{[a,b]}(E) \implies x \in \mathcal{R}^*_{[a,b]}(E)$ is trivial.

To see that the opposite implication holds, take $x \in \mathcal{R}^*_{[a,b]}(E)$ and set $w(t) = \dot{x}(t)$ and $m(t,u) = f(t, x(t), u)$.

Assumption (LC) together with Proposition 2.3.4 in [22] guarantees that $(t,u) \to m(t,u)$ is $\mathcal{L}\times\mathcal{B}$ measurable. Set set $G(t) = S_m(t, x(t))$. We have $w(t) \in \{m(t,u) : \ u \in G(t)\}$ for almost every $t \in [a,b]$. Since G is a measurable and closed set-valued mapping, Theorem 2.3.13 in [22] asserts the existence of a measurable function $u : [a,b] \to \mathbb{R}^k$ such that

$$u(t) \in G(t) \quad \text{a.e.} \quad \text{and} \quad w(t) = m(t, u(t)) \quad \text{a.e.}$$

It follows that $x \in \mathcal{S}^*_{[a,b]}(E)$, completing our proof. □

It is worth mentioning that the previous result does not require convexity of $F_m(t,x)$ (or $S_m(t,x)$). If however (CA) holds, then it can easily be seen that Theorem 2.6.1 in [22] can be applied (by Lemma 6.1 and Corollary 6.4) leading to the following.

Theorem 7.3 *Assume that x^* defining X satisfies (7), f satisfies (B1) and (LC) and (B2) – (B4), (BS) and (CA) hold. Then $\mathcal{R}^*_{[a,b]}(E)$ is compact with respect to the supremum norm topology.*[2]

8 Applications and Future Work

An straightforward consequence of our previous results is the existence of solution to some optimal control problems with mixed constraints. Let us consider the problem

$$(Q) \quad \begin{cases} \text{Minimize } l(x(b)) \\ \text{subject to} \\ \dot{x}(t) = f(t, x(t), u(t)) & \text{a.a. } t \in [a,b] \\ (x(t), u(t)) \in S(t) & \text{a.a. } t \in [a,b] \\ (x(a), x(b)) \in \{x_a\} \times E_b \end{cases}$$

where $l : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function and E_b a closed set. Assume that f satisfies (B1) and (LC) and that (B2)-(B4) as well as (BS) and (CA) hold. Let us suppose that there exists a feasible process (x^*, u^*) for (Q) and consider some $\varepsilon > 0$. Define as before $X(t) = x^*(t) + \varepsilon\bar{B}$. Then Theorem 7.2 allows us to appeal to Proposition 2.6.2 in [22], to deduce existence of a minimizer for (Q).

When $S(t)$ is as defined by (4), existence of solution for (Q) is proved in, for example, [5], assuming that the function $u \to f(t,x,u)$ as well as $u \to g(t,x,u)$ are convex. Our approach covers a different class of problems, those for which (CA) holds. Recall that convexity of $u \to g(t,x,u)$ implies convexity of $S_m(t,x)$ but convexity of $S_m(t,x)$ does not necessarily implies convexity of $F_m(t,x)$ even when the components of f are convex functions with respect to u

If (CA) does not hold, relaxation results along the lines of Proposition 2.7.3 in [22] are valid. In this respect, and as expected, the set-valued mapping $\operatorname{co} F_m(t,x)$ plays an important rule. But some care needs to be

[2]Once again, we may replace the requirement that x^* defining X satisfies (7) by an integrable bound on the values of $f(t,x,u)$ when $x \in X(t)$.

considered when the convex hull of $F_m(t,x)$ is taken. For example, when $S(t)$ is given by (4), we can translate the convexity of $F_m(t,x)$ to (Σ), appealing to Carathéodory's Theorem, leading to

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n+1} \lambda_i(t) f(t, x(t), u_i(t)), \quad \text{a.e.,} \\ g(t, x(t), u_i(t)) \leqslant 0, \; i = 1, \ldots, n+1, \quad \text{a.e.,} \\ (\lambda_1(t), \ldots, \lambda_{n+1}(t)) \in \Lambda, \quad \text{a.e.} \\ u_i(t) \in U, \quad \text{a.e. for } i = 1, \ldots, n+1 \end{cases} \tag{17}$$

where

$$\Lambda := \{\lambda' \in \mathbb{R}^{n+1} : \lambda' \geqslant 0 \text{ and } \sum_{i=1}^{n+1} \lambda'_i = 1\}.$$

A word of caution when a running cost is added to the cost in (Q), i. e., when we consider the problem

$$(\mathrm{Q}') \quad \begin{cases} \text{Minimize } l(x(b)) + \int_a^b L(t, x(t), u(t)) dt \\ \text{subject to} \\ \qquad \dot{x}(t) = f(t, x(t), u(t)) \qquad \text{a.a. } t \in [a,b] \\ \qquad (x(t), u(t)) \in S(t) \qquad \text{a.a. } t \in [a,b] \\ \qquad (x(a), x(b)) \in \{x_a\} \times E_b \end{cases}$$

As it is commonly done, assume that L satisfies the same conditions f satisfies, in this case, (B1) and (LC). Define

$$\widetilde{F}_m(t,x) = \{(f(t,x,u), L(t,x,u)) : \; u \in S_m(t,x)\}$$

and

$$V_L(t,x) = \{(v,y) : \; v = f(t,x,u), \; y \geqslant L(t,x,u)) : \; u \in S_m(t,x)\}.$$

Clearly we have

$$\mathrm{co}\, \widetilde{F}_m(t,x) \subseteq \mathrm{co}\, V_L(t,x)$$

but, in general, convexity of $V_L(t,x)$ does not imply that of $F_m(t,x)$ as it shown in Chapter IV of [17].

The reformulation of the systems similar to (Σ) into differential inclusions can be useful to derive necessary conditions of optimality when combined with extended Euler-Lagrange conditions in the vein of [22] and [9] (see also reference within). Following closely [9] this is an important tools in [10] and [11], two papers where necessary conditions for mixed constrained optimal control problems are extensively studied.

The consequences of the the second part of Lemma 6.5 asserting that under some conditions, any control of our system maybe a feed back control depending Lipschitz continuously on the state deserves further investigation. In particular, the derivation of necessary conditions along the lines of [1] may be of use.

Acknowledgements The authors would like to thank Prof. Hasnaa Zidani for pushing us to clarify many of the subjects treated here.

The financial support of FCT projects PTDC/EEA-ELC/122203/2010 and PTDC/EEI-AUT/1450/2012|FCOMP-01-0124-FEDER-028894, and European Union FP7 (FP7-PEOPLE-2010-ITN, Grant Agreement no. 264735-SADCO) are gratefully acknowledged.

Bibliography

[1] Artstein, Z. 2011. *Pontryagin Maximum Principle revisited with feedbacks.* European Journal of Control 17, pp. 46–54

[2] Aubin J. P., Cellina, A. 1984. *Differential Inclusions: Set-Valued Maps and Viability Theory* Springer-Verlag.

[3] Aubin, J. P., Frankowska, H. 1990. *Set-Valued Analysis.* Birkhäuser, Boston.

[4] Bokanowski O., Cristiani E., Laurent-Varin J., Zidani H. 2012. *Hamilton-Jacobi-Bellman approach for the climbing problem for multi-stage launchers.* Proceedings of MTNS 2012,July 2012, Melbourne, Australia.

[5] Cesari, L. 1983 *Optimization-theory and applications: problems with ordinary differential equations* Springer-Verlag.

[6] Clarke, F. 1976. *The maximum principle under minimal hypotheses.* SIAM J. Control Optim. 14, pp. 1078-1091.

[7] Clarke F. 1983. *Optimization and Nonsmooth Analysis.* John Wiley, New York.

[8] Clarke F., Ledyaev Yu. S.,Stern R. J., Wolenski P. R. 1998. *Nonsmooth Analysis and Control Theory.* Springer-Verlag, New York.

[9] Clarke F. 2005. *Necessary conditions in dynamic optimization.* Mem. Amer. Math. Soc.

[10] Clarke F., de Pinho M. d. R. 2010. *Optimal control problems with mixed constraints.* SIAM J. Control Optim. 48, pp. 4500–4524.

[11] Clarke F., Ledyaev Y., de Pinho M. d. R. 2011. *An extension of the schwarzkopf multiplier rule in optimal control.* SIAM J. Control Optim. 49, pp. 599–610.

[12] Clarke F. 2013. *Functional Analysis, Calculus of Variations and Optimal Control.* Springer, Graduate Texts in Mathematics, Vol. 267.

[13] Devdaryani E. N., Ledyaev Y. S. 1999. *Maximum principle for implicit control systems*, Appl. Math. Optim., 40, 79–103.

[14] Folland, G. B., 1999. *Real Analysis, Modern Techniques and Their Applications.* Pure and Applied Mathematics, John Wiley & Sons, New York.

[15] Kornienko I., de Pinho, MdR, 2013. *Properties of control systems with mixed constraints in the form of inequalities.* Internal Report, ISR, DEEC, FEUP, 2013.

[16] Loewen P, Rockafellar R. T., 1994. *Optimal Control of Unbounded Differential Inclusions* SIAM J. Control Optim. Vol. 32, No. 2, pp. 442–470.

[17] Macki, J. and Strauss, A. 1981. *Introduction to Optimal Control Theory.* Springer.

[18] Mordukhovich B. 2006. *Variational analysis and generalized differentiation. Basic Theory.* Fundamental Principles of Mathematical Sciences 330, Springer-Verlag, Berlin.

[19] Miller Neilan, R. L., Lenhart S. 2010. *An Introduction to Optimal Control with an Application in Disease Modeling.* DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume 75, pp. 67–81.

[20] Rockafellar R. T. and Wets B. 1998. *Variational Analysis*, Grundlehren Math. Wiss. 317, Springer-Verlag, Berlin.

[21] Smirnov, G. V. 2002 *Introduction to the Theory of Differential Inclusions*. American Mathematical Soc.

[22] Vinter R. 2000. *Optimal Control*. Birkhäuser, Boston.