Chapter 4
Duality in Linear Programming

Companion slides of
Applied Mathematical Programming
by Bradley, Hax, and Magnanti
(Addison-Wesley, 1977)

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What is duality?

- Shadow prices = optimal simplex multipliers
  – Marginal worth of an additional u nit of resource
  – Opportunity costs of resource allocation when pricing out a new activity
- Duality is
  – a unifying theory that develops the relationships between a given linear program and another related linear program stated in terms of variables with this shadow-price interpretation.

This unified theory is important

1. Because it allows fully understanding the shadow-price interpretation of the optimal simplex multipliers, which can prove very useful in understanding the implications of a particular linear-programming model.
2. Because it is often possible to solve the related linear program with the shadow prices as the variables in place of, or in conjunction with, the original linear program, thereby taking advantage of some computational efficiencies.

A PREVIEW OF DUALITY

A preview of duality

Firm producing three types of automobile trailers:

x_1 = number of flat-bed trailers produced per month,

x_2 = number of economy trailers produced per month,

x_3 = number of luxury trailers produced per month.

The constraining resources of the production operation are the metalworking and woodworking capacities measured in days per month.

The linear program to maximize contribution to the firm's overhead (in hundreds of dollars) is:

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$
\begin{aligned}
\tfrac{1}{2}x_1 + \ 2x_2 + \quad x_3 &\le 24, \\
x_1 + \ 2x_2 + \ 4x_3 &\le 60, \\
x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 &\ge 0.
\end{aligned}
$$

Initial tableau in canonical form

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	2	1	1	
x_5	60	1	2	4		1
$(-z)$	0	6	14	13		

Final (optimal) tableau

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

- The shadow prices, y_1 for metalworking capacity and *y2* for woodworking capacity, can be determined from the final tableau as the negative of the reduced costs associated with the slack variables *x4* and *x5.*
- Thus these shadow prices are *y1 = 11* and *y2 = 1/2*, respectively.

Economic properties of the shadow prices associated with the resources

Reduced costs in terms of shadow prices:

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad (j = 1, 2, \ldots, n).$$

Since a_{ij} is the amount of resource i used per unit of activity j, and y_i is the imputed value of that resource, the term

$$\sum_{i=1}^{m} a_{ij} y_i$$

is the total value of the resources used per unit of activity j.

$$\sum_{i=1}^{m} a_{ij} y_i$$

is thus the marginal resource cost for using activity *j*.

If we think of the objective coefficients c_j as being marginal revenues, the reduced costs

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i$$

are simply net marginal revenues (i.e., marginal revenue minus marginal cost).

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

- For the basic variables the reduced costs are zero. The values imputed to the resources are such that the net marginal revenue is zero on those activities operated at a positive level. That is, for any production activity at positive level, marginal revenue must equal marginal cost.

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

- The reduced costs for all nonbasic variables are negative. The interpretation is that, for the values imputed to the scarce resources, marginal revenue is less than marginal cost for these activities, so they should not be pursued.

Shadow prices are interpreted as the opportunity costs associated with consuming the firm's resources

If we value the firm's total resources at the shadow prices, we find their value

$$v = 11(24) + \tfrac{1}{2}(60) = 294,$$

is exactly equal to the optimal value of the objective function of the firm's decision problem.

Can shadow prices be determined directly, without solving the firm's production-decision problem?

- The shadow prices must satisfy the requirement that marginal revenue be less than or equal to marginal cost for all activities.
- Further, they must be nonnegative since they are associated with less-than-or-equal-to constraints in a maximization decision problem.

$$
\begin{aligned}
\tfrac{1}{2}y_1 + y_2 &\geq 6, \\
2y_1 + 2y_2 &\geq 14, \\
y_1 + 4y_2 &\geq 13, \\
y_1 \geq 0, \qquad y_2 &\geq 0.
\end{aligned}
$$

Imagine that the firm does not own its productive capacity but has to rent it…

Shadow prices = rent rates

$$\text{Minimize } v = 24y_1 + 60y_2,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}y_1 + y_2 &\geq 6, \\ 2y_1 + 2y_2 &\geq 14, \\ y_1 + 4y_2 &\geq 13, \end{aligned}$$

$$y_1 \geq 0, \qquad y_2 \geq 0.$$

optimal solution is *y1 = 11, y2 = 1*/2 , and v = 294.

By solving (2), we can determine the shadow prices of (1) directly

(1)

Maximize $z = 6x_1 + 14x_2 + 13x_3,$

$$\begin{aligned} \tfrac{1}{2}x_1 + \ 2x_2 + \ \ x_3 &\le 24, \\ x_1 + \ 2x_2 + 4x_3 &\le 60, \end{aligned}$$

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

(2)

Minimize $v = 24y_1 + 60y_2,$

$$\begin{aligned} \tfrac{1}{2}y_1 + \ \ y_2 &\ge \ 6, \\ 2y_1 + 2y_2 &\ge 14, \\ y_1 + 4y_2 &\ge 13, \end{aligned}$$

$$y_1 \ge 0, \qquad y_2 \ge 0.$$

PRIMAL

DUAL

Solving the dual by the simplex method solves the primal as well

- The decision variables do the dual give the shadow prices of the primal.

  y1 = 11, y2 = 1/2, and v = 294

- The shadow prices of the dual give the decision variables of the primal.

  $u_1 = 36$, $u_2 = 0$, and $u_3 = 6$

Implications of solving these problems by the simplex method

- The optimality conditions of the simplex method require that the reduced costs of basic variables be zero, i.e.:

$$\begin{aligned}
&\text{if } \hat{x}_1 > 0, \quad \text{then } \overline{c}_1 = 6 - \tfrac{1}{2}\hat{y}_1 - \hat{y}_2 = 0; \\
&\text{if } \hat{x}_3 > 0, \quad \text{then } \overline{c}_3 = 13 - \hat{y}_1 - 4\hat{y}_2 = 0.
\end{aligned}$$

(if a decision variable of the primal is positive, then the corresponding constraint in the dual must hold with equality)

Implications of solving these problems by the simplex method

- The optimality conditions require that the nonbasic variables be zero (at least for those variables with negative reduced costs), i.e.:

$$\text{if } \overline{c}_2 = 14 - 2\hat{y}_1 - 2\hat{y}_2 < 0, \quad \text{then } \hat{x}_2 = 0.$$

(if a constraint holds as a strict inequality, then the corresponding decision variable must be zero)

Implications of solving these problems by the simplex method (chapter 3)

- If some shadow price is positive, then the corresponding constraint must hold with equality, i.e.:

$$\begin{aligned}
&\text{if } \hat{y}_1 > 0, \quad \text{then } \tfrac{1}{2}\hat{x}_1 + 2\hat{x}_2 + \hat{x}_3 = 24; \\
&\text{if } \hat{y}_2 > 0, \quad \text{then } \hat{x}_1 + 2\hat{x}_2 + 4\hat{x}_3 = 60.
\end{aligned}$$

- If a constraint of the primal is not binding, then its corresponding shadow price must be zero

Complementary slackness conditions

- If a variable is positive, its corresponding (complementary) dual constraint holds with equality.
- If a dual constraint holds with strict inequality, then the corresponding (complementary) primal variable must be zero.

FINDING THE DUAL IN GENERAL

DEFINITION OF THE DUAL PROBLEM

Primal-Dual formalization

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Generalization

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

$$\text{Minimize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y_i' + \sum_{i=m''+1}^{m} b_i y_i'',$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$x_j \ge 0$$

$$\begin{aligned} \sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y_i' + \sum_{i=m''+1}^{m} a_{ij} y_i'' &\ge c_j \\ y_i &\ge 0, \\ -y_i' &\ge 0, \end{aligned}$$

Generalization

$$\text{Minimize } z = \sum_{j=1}^{n} c_j x_j,$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$x_j \geq 0$$

$$\text{Maximize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y_i' + \sum_{i=m''+1}^{m} b_i y_i'',$$

$$\sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y_i' + \sum_{i=m''+1}^{m} a_{ij} y_i'' \geq c_j$$

$$y_i \leq 0 \qquad (i = 1, 2, \ldots, m'),$$

$$y_i' \geq 0 \qquad (i = m' + 1, m' + 2, \ldots, m'')$$

Generalization

Primal (Maximize)	*Dual (Minimize)*
ith constraint $\leq$	ith variable ≥ 0
ith constraint $\geq$	ith variable ≤ 0
ith constraint $=$	ith variable unrestricted
jth variable ≥ 0	jth constraint $\geq$
jth variable ≤ 0	jth constraint $\leq$
jth variable unrestricted	jth constraint $=$

THE FUNDAMENTAL DUALITY PROPERTIES

A little bit of theory…

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Weak Duality Property

If $\overline{x}_j,\ j = 1, 2, \ldots, n,$

is a feasible solution to the primal problem and

$\overline{y}_i,\ i = 1, 2, \ldots, m,$

is a feasible solution to the dual problem, then

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i .$$

Dual feasible	$\downarrow$	v decreasing
Primal feasible	$\uparrow$	z increasing

Optimality Property

If $\hat{x}_j,\ j = 1, 2, \ldots, n,$

is a feasible solution to the primal problem and

$\hat{y}_i,\ i = 1, 2, \ldots, m,$

is a feasible solution to the dual problem, then

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i,$$

and they are both optimal for their problems.

Unboundedness Property

- If the primal (dual) problem has an unbounded solution, then the dual (primal) problem is infeasible.

Strong Duality Property

- If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem, and these two values are equal.

THE COMPLEMENTARY SLACKNESS

Complementary Slackness Property

- If, in an optimal solution of a linear program, the value of the dual variable (shadow price) associated with a constraint is nonzero, then that constraint must be satisfied with equality.
- Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

Complementary Slackness Property

- For the primal linear program posed as a maximization problem with less-than-or-equal-to constraints, this means:

$$\text{i)} \qquad \text{if } \hat{y}_i > 0, \quad \text{then } \sum_{j=1} a_{ij}\hat{x}_j = b_i;$$

$$\text{ii)} \qquad \text{if } \sum_{j=1}^{n} a_{ij}\hat{x}_j < b_i, \quad \text{then } \hat{y}_i = 0.$$

Complementary Slackness Property

- For the dual linear program posed as a minimization problem with greater-than-or-equal-to constraints, this means:

$$\text{iii)} \quad \text{if } \hat{x}_j > 0, \quad \text{then } \sum_{i=1}^{m} a_{ij} y_i = c_j,$$

$$\text{iv)} \quad \text{if } \sum_{i=1}^{m} a_{ij} \hat{y}_i > c_j, \quad \text{then } \hat{x}_j = 0.$$

Optimality Conditions

If $\hat{x}_j, \ j = 1, 2, \ldots, n,$

and $\hat{y}_i, i = 1, 2, \ldots, m,$

are feasible solutions to the primal and dual problems, respectively,
then they are optimal solutions to these problems if, and only if,
the complementary-slackness conditions hold for both the primal and the dual problems.

THE DUAL SIMPLEX METHOD

The Dual Simplex Method

- Solving the primal problem, moving through solutions (simplex tableaus) that are dual feasible but primal unfeasible.(1)
  - Primal feasible: $\overline{b}_i \geq 0$
  - Dual feasible: $\overline{c}_j \leq 0$

- An optimal solution is a solution that is both primal and dual feasible.

(1) This is different from Solving the dual problem with the (primal) simplex method…

The rules of the dual simplex method are identical to those of the primal simplex algorithm

- Except for the selection of the variable to leave and enter the basis.
- At each iteration of the dual simplex method, we require that:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} y_i a_{ij} \le 0;$$

  and since $y_i \ge 0$ for $i = 1, 2, \ldots, m,$
  these variables are a dual feasible solution.
- Further, at each iteration of the dual simplex method, the most negative $\overline{b}_i$ is chosen to determine the pivot row, corresponding to choosing the most positive $\overline{c}_j$ to determine the pivot column in the primal simplex method.

Example

Dual feasible initial solution:

Basic variables	*Current values*	x_1	x_2	x_3	x_4
x_3	-1	-1	-1	1	
x_4	-2	-2	-3		1
$(-z)$	0	-3	-1		

1st

2nd

$$\underset{i}{\text{Min}}\ \{\overline{b}_i\} = \text{Min}\ \{-1, -2\}$$

(most negative)

$$\underset{j}{\text{Min}} \left\{ \frac{\overline{c}_j}{\overline{a}_{rj}} \middle| \overline{a}_{rj} < 0 \right\} = \text{Min} \left\{ \frac{-3}{-2}, \frac{-1}{-3} \right\}$$

Example

Basic variables	*Current values*	x_1	x_2	x_3	x_4
x_3	$-\frac{1}{3}$	$-\frac{1}{3}$		1	$-\frac{1}{3}$
x_2	$\frac{2}{3}$	$\frac{2}{3}$	1		$-\frac{1}{3}$
$(-z)$	$\frac{2}{3}$	$-\frac{7}{3}$			$-\frac{1}{3}$

Example

Basic variables	*Current values*	x_1	x_2	x_3	x_4
x_4	1	1		-3	1
x_2	1	1	1	-1	
$(-z)$	1	-2		-1	

Optimal solution!

PRIMAL-DUAL ALGORITHMS

Primal-dual algorithms

- Algorithms that perform both primal and dual steps, e.g., *parametric primal-dual algorithm.*

An example of application of the the parametric primal-dual algorithm

$$
\begin{aligned}
x_1 &\geq 0, \qquad x_2 \geq 0, \\
x_1 + x_2 &\leq 6, \\
-x_1 + 2x_2 &\leq -\tfrac{1}{2}, \\
x_1 - 3x_2 &\leq -1, \\
-2x_1 + 3x_2 &= z(\max).
\end{aligned}
$$

- After adding slack variables (canonical form) it is not feasible, neither primal or dual.
- We will consider an arbitrary parameter θ large enough, so that this system of equations satisfies the primal feasibility and primal optimality conditions.

Example

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_3	6	1	1	1		
x_4	$-\frac{1}{2}+\theta$	-1	2		1	
x_5	$-1+\theta$	1	-3			1
$(-z)$	0	-2	$(3-\theta)$			

- Decrease θ until zero, performing the needed iterations, primal or dual, depending on where unfeasibility appears.

Example

$\theta < 3$.

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_3	6	1	1	1		
x_4	$-\frac{1}{2} + \theta$	-1	2		1	
x_5	$-1 + \theta$	1	-3			1
$(-z)$	0	-2	$(3 - \theta)$			

Example

$$\frac{7}{10} \le \theta \le 3$$

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	$6\frac{1}{4} - \frac{1}{2}\theta$	$\frac{3}{2}$		1	$-\frac{1}{2}$	
x_2	$-\frac{1}{4} + \frac{1}{2}\theta$	$-\frac{1}{2}$	1		$\frac{1}{2}$	
$\leftarrow$ x_5	$-\frac{7}{4} + \frac{5}{2}\theta$	$-\frac{1}{2}$			$\frac{3}{2}$	1
$(-z)$	$-(3-\theta)(-\frac{1}{4} + \frac{1}{2}\theta)$	$(-\frac{1}{2} - \frac{1}{2}\theta)$ $\uparrow$			$(-\frac{3}{2} + \frac{1}{2}\theta)$	

Example

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_3	$1+7\theta$			1	4	3
x_2	$\frac{3}{2}-2\theta$		1		-1	-1
x_1	$\frac{7}{2}-5\theta$	1			-3	-2
$(-z)$	$-(3-\theta)\left(-\frac{1}{4}+\frac{1}{2}\theta\right)$ $+(\frac{1}{2}+\frac{1}{2}\theta)(\frac{7}{2}-5\theta)$				$(-3-\theta)$	$(-1-\theta)$

- As we continue to decrease θ to zero, the optimality conditions remain satisfied. Thus the optimal final tableau for this example is given by setting θ equal to zero.

MATHEMATICAL ECONOMICS

The dual as the price-setting mechanism in a perfectly competitive economy

- Suppose that a firm may engage in any *n* production activities that consume and/or produce *m* resources in the process.
- Let:
  - $x_j \geq 0$ be the level at which the *j*th activity is operated,
  - c_j be the revenue per unit (minus means cost) generated from engaging in the *j*th activity,
  - a_{ij} be the amount of the *i*th resource consumed (minus means produced) per unit level of operation of the *j*th activity.
- Assume that the firm starts with a position of b_i units of the *i*th resource and may buy or sell this resource at a price
  $y_i \geq 0$ determined by an external market.

- Since the firm generates revenues and incurs costs by engaging in production activities and by buying and selling resources, its profit is given by:

$$\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

where the second term includes revenues from selling excess resources and costs of buying additional resources.

If $b_i > \sum_{j=1}^{n} a_{ij} x_j$

the firm sells $b_i - \sum_{j=1}^{n} a_{ij} x_j$

units of resources *i* to the marketplace at a price y_i .

If $b_i < \sum_{j=1}^{n} a_{ij} x_j$

the firm buys $\sum_{j=1}^{n} a_{ij} x_j - b_i$

units of resource i from the marketplace at a price y_i .

The “malevolent market"

- The market mechanism for setting prices is such that it tends to minimize the profits of the firm, since these profits are construed to be at the expense of someone else in the market.
- That is, given x_j for $j = 1, 2, \ldots, n$, the market reacts to minimize the firm’s profit:

$$\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

With this market…

- Consuming any resource that needs to be purchased from the marketplace, is clearly uneconomical for the firm, since the market will tend to set the price of the resource arbitrarily high so as to make the firm’s profits arbitrarily small.
- Therefore, the firm will choose to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m).$$

- If any resource were not completely consumed by the firm in its own production and therefore became available for sale to the market, this ‘‘malevolent market" would set a price of zero for that resource in order, again, to minimize the firm’s profit.
- Therefore, the profit expression reduces to:

$$\sum_{j=1}^{n} c_j x_j$$

In short, in a ‘‘malevolent market“ the firm’s problem is:

$$\text{Maximize} \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

And now the impact of the firm's decisions on the market…

- Rearranging the profit expression:

$$\sum_{j=1}^{n}\left(c_j - \sum_{i=1}^{m} a_{ij} y_i\right) x_j + \sum_{i=1}^{m} b_i y_i$$

Look at the problem from the standpoint of the market…

- If the market sets the prices so that the revenue from engaging in an activity exceeds the market cost, then the firm would be able to make arbitrarily large profits by engaging in the activity at an arbitrarily high level, a clearly unacceptable situation from the standpoint of the market.
- The market instead will always choose to set its prices such that:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n).$$

- If the market sets the price of a resource so that the revenue from engaging in that activity does not exceed the potential revenue from the sale of the resources directly to the market, then the firm will not engage in that activity at all. In this case, the opportunity cost associated with engaging in the activity is in excess of the revenue produced by engaging in the activity.
- Therefore, the profit expression reduces to:

$$\sum_{i=1}^{m} b_i y_i$$

In short, the market's "decision" problem, of choosing the prices for the resources so as to minimize the firm's profit, reduces to:

$$\text{Minimize} \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

What is the relationship between these solutions?

- The firm and the market would interact in such a manner that an equilibrium would be arrived at, satisfying:

$$\left(b_i - \sum_{j=1}^{n} a_{ij}\hat{x}_j\right)\hat{y}_i = 0 \qquad (i = 1, 2, \ldots, m),$$

$$\left(c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\hat{x}_j = 0 \qquad (j = 1, 2, \ldots, n).$$

Complementary-slackness conditions!

$$\left(b_i - \sum_{j=1}^{n} a_{ij}\hat{x}_j\right)\hat{y}_i = 0$$

- If a firm has excess of a particular resource, then the market should not be willing to pay anything for the surplus of that resource since the market wishes to minimize the firm’s profit.
- There may be a nonzero market price on a resource only if the firm is consuming all of that resource that is available.

$$\left(c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\hat{x}_j = 0$$

- Either the amount of excess profit on a given activity is zero or the level of that activity is zero.
- That is, a perfectly competitive market acts to eliminate any excess profits.

If we had an equilibrium satisfying the complementary-slackness conditions…

Then

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i$$

the maximum profit that the firm can make equals the market evaluation of its initial endowment of resources.

In a perfectly competitive market
no firm makes excess profits.

GAME THEORY

The perfectly competitive economy as a game between the firm and the malevolent market

- The firm chooses its strategy to maximize its profits while the market behaves (“chooses” its strategy) in such a way as to minimize the firm’s profits.
- Duality theory is, in fact, closely related to game theory.

In many contexts, a decision-maker does not operate in isolation, but in contend with other decision-makers with conflicting objectives

- Game theory is one approach for dealing with these ‘‘multiperson" decision problems.
- It views the decision-making problem as a *game* in which each decision-maker, or *player*, chooses a *strategy* or an action to be taken.
- When all players have selected a strategy, each individual player receives a *payoff*.

Example

- There are two players firm R (row player) and firm C (column player).
- The alternatives open to each firm are its advertising possibilities.
- Payoffs are market shares resulting from the combined advertising selections of both firms.

Market share of firm R

Firm R alternatives \ Firm C alternatives	*Advertising campaign 1*	*Advertising campaign 2*	*Advertising campaign 3*
Advertising campaign 1	30%	40%	60%
Advertising campaign 2	20%	10%	30%

- Since we have assumed a two-firm market, firm R and firm C share the market, and firm C receives whatever share of the market R does not.
- Consequently, firm R would like to maximize the payoff entry from the table and firm B would like to minimize this payoff.

Games with this structure are called *two-person, zero-sum games.*
They are *zero-sum,* since the gain of one player is the loss of the other player.

Behavioral assumptions as to how the players will act

- Both players are conservative, in the sense that they wish to assure themselves of their possible payoff level regardless of the strategy adopted by their opponent.
- It selecting its alternative, firm R chooses a row in the payoff table. The worst that can happen from its viewpoint is for firm C to select the minimum column entry in that row.

Firm R

Firm R alternatives \ Firm C alternatives	Advertising campaign 1	Advertising campaign 2	Advertising campaign 3
Advertising campaign 1	30%	40%	60%
Advertising campaign 2	20%	10%	30%

- If firm R selects its first alternative, then it can be assured of securing 30% of the market.
- If it selects its second alternative it is assured of securing 10%, but no more.
- It will chose alternative 1: *maxmin* strategy

Firm C

Firm R alternatives \ Firm C alternatives	*Advertising campaign 1*	*Advertising campaign 2*	*Advertising campaign 3*
Advertising campaign 1	30%	40%	60%
Advertising campaign 2	20%	10%	30%

- If firm C selects its first alternative, then it can be assured of losing only 30% of the market, but no more.
- If it selects its second alternative it is assured of losing 40%, but no more.
- If it selects its third alternative it is assured to lose 60%, but no more.
- It will chose alternative 1: *minmax* strategy

The payoff is the same for both players' best decision

- The problem has a *saddlepoint,* an *equilibrium* solution
  - Neither player will move unilaterally from this point.
  - When firm R adheres to alternative 1, then firm C cannot improve its position by moving to either alternative 2 or 3 since then firm R's market share increases to either 40% or 60%.
  - Similarly, when firm C adheres to alternative 1, then firm R as well will not be induced to move from its saddlepoint alternative, since its market share drops to 20% if it selects alternative 2.

A new payoff table…
(firm R market share)

Firm R \ *Firm C*	*Alternative 1*	*Alternative 2*	*Alternative 3*	*Security level for firm R*
Alternative 1	30	40	60	30
Alternative 2	60	10	30	10
Security level for firm C	60	40	60	

maximin = 30

minimax = 40

- Given any choice of decisions by the two firms, one of them can always improve its position by changing its strategy.
- If, for instance, both firms choose alternative 1…

And if a player selects a strategy according to some preassigned probabilities, instead of choosing outright?

- Suppose that firm C selects among its alternatives with probabilities x_1, x_2, and x_3, respectively.
- Then the expected market share of firm R is:

$$30x_1 + 40x_2 + 60x_3$$

if firm R selects alternative 1, or

$$60x_1 + 10x_2 + 30x_3$$

- if firm R selects alternative 2.

- Since any gain in market share by firm R is a loss to firm C, firm C wants to make the expected market share of firm R as small as possible.
- Firm C can minimize the maximum expected market share of firm R by solving the following linear program:

$$\text{Minimize } v,$$

subject to:

$$\begin{aligned}
30x_1 + 40x_2 + 60x_3 - v &\le 0,\\
60x_1 + 10x_2 + 30x_3 - v &\le 0,\\
x_1 + x_2 + x_3 &= 1,
\end{aligned}$$

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

- The solution to this linear program is:

$$x_1 = \frac{1}{2},\ x_2 = \frac{1}{2},\ x_3 = 0, \text{ and } v = 35.$$

  with an expected market share for firm R decreasing from 40 percent (minimax strategy payoff) to 35 percent.

Let us now see how firm R might set probabilities y_1 and y_2 on its alternative selections to achieve its best security level

- When firm R weights its alternatives by y_1 and y_2, it has an expected market share of:

$$\begin{array}{ll} 30y_1 + 60y_2 & \text{if firm C selects alternative 1,} \\ 40y_1 + 10y_2 & \text{if firm C selects alternative 2,} \\ 60y_1 + 30y_2 & \text{if firm C selects alternative 3.} \end{array}$$

- Firm R wants its market share as large as possible, but takes a conservative approach in maximizing its minimum expected market share from these three expressions.
- In this case, firm R solves the following linear program:

$$\text{Maximize } w,$$

subject to:

$$\begin{aligned}
30y_1 + 60y_2 - w &\geq 0,\\
40y_1 + 10y_2 - w &\geq 0,\\
60y_1 + 30y_2 - w &\geq 0,\\
y_1 + y_2 &= 1,\\
y_1 \geq 0, \qquad y_2 &\geq 0.
\end{aligned}$$

- Firm R acts optimally by selecting its first alternative with probability $y_1 = 5/6$ and its second alternative with probability $y_2 = 1/6$, giving an expected market share of 35 percent.

The security levels resulting from each linear program are identical!

- The linear programming problems are dual of each other (check…).

- Two-person, zero-sum games reduce to primal and dual linear programs!

Historically, game theory was first developed by John von Neumann in 1928 and then helped motivate duality theory in linear programming some twenty years later.