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ABSTRACT
The laminar flat plate boundary-layer flow solution for finitely extensible nonlinear elastic model with Peterlin’s closure fluids, originally derived by Olagunju [D. O. Olagunju, Appl. Math. Comput. 173, 593–602 (2006)], is revisited by relaxing some of the assumptions related to the conformation tensor. The ensuing simplification through an order of magnitude analysis and the use of similarity-like variables allows for a semi-analytical approximate similarity solution to be obtained. The proposed solution is more accurate than the original solution, and it tends to self-similar behavior only in the limit of low elasticity. Additionally, we provide a more extensive set of results, including profiles of polymer conformation tensor components, laws of decay for peak stresses and their location, as well as the streamwise variations of boundary layer thickness, displacement and momentum thicknesses. We also provide asymptotic laws for these quantities under low elasticity flow conditions. Comparisons with results from the numerical solution of the full set of governing equations show the approximate similarity solution to be valid up to a high local Weissenberg number \( \text{Wi} \) between 0.2 and 0.3, corresponding to a local Weissenberg number based on the boundary layer thickness of about 10, for a wide range of values of dumbbell extensibility and solvent viscosity ratio. Above this critical condition, the semi-analytical solution is unable to describe the complex variations of the conformation tensor within the boundary layer, but it still remains accurate in its description of the velocity profiles, friction coefficient, and the variations of displacement, momentum, and boundary layer thicknesses for Weissenberg numbers at least one order of magnitude higher (within 5% up to \( \text{Wi} \approx 5 \)).

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I. INTRODUCTION
Boundary layer flows are ubiquitous in industrial and natural flows in such a variety of conditions that more than 100 years after the seminal work of Prandtl1 it continues to be the source of relevant research, as in Refs. 2–4. The topic has been far less investigated for fluids of non-Newtonian rheology and, in particular, when they exhibit viscoelastic characteristics. One particular exception is the turbulent pipe or channel flow of solutions of polymers and worm-like micelles. Toms’s finding,7 that the minor addition of polymer molecules to a Newtonian solvent imparts viscoelastic fluid properties that induce severe drag reduction in turbulent flow, fostered a wealth of experimental,8 theoretical,7,9 and numerical,10 research that continues up to this day, as reviewed recently by White and Mungal.10

The more fundamental laminar boundary layer flow over a flat plate remains less investigated for viscoelastic fluids and is the topic of this work. These fluids are incompressible liquids, and Srivastava11 and Rajeswari and Rathna12 proposed approximate solutions by utilizing the Kármán-Pohlhausen method13,14 to study the boundary layer flows of second order Rivlin–Ericksen fluids in the vicinity of a stagnation point. Later and inspired by Prandtl’s theory, Beard and Walters15 proposed a similarity solution for boundary layer flows of viscoelastic Oldroyd-B fluids near a stagnation point and reported that increasing the elasticity increases the velocity inside the boundary-layer and the wall stress. In 1982, Rajagopal et al.16 studied Falkner–Skan flows of second order fluids and concluded that this type of solution requires not only a large Reynolds number \( \text{Re} \), but also that \( \text{Re}/\text{Wi} \gg 1 \), where \( \text{Wi} \) is the Weissenberg number, the ratio between elastic and viscous forces. In contrast with Newtonian fluids, which follow a single rheological constitutive equation, non-Newtonian fluids are characterized by a wide range of behaviors, and this may be accompanied by a variety of rheological constitutive equations.17 Early constitutive models, as used in the above works, were essentially derived from continuum mechanics, which described in an incomplete form some of the
relevant non-Newtonian fluid properties (Rivlin–Eriksen and second order fluids may not be even valid on account of the large deformation rates encountered in boundary layer flows). The advent of structural or kinetic-theory based constitutive models has resulted in better qualitative and in some cases quantitative, descriptions of the rheology of real fluids. A well-known example of the former is the successful description of the rheology of some polymer melts by the Phan-Thien–Tanner model.\textsuperscript{9,20}

Paradoxically, the description of dilute polymer solutions has remained more elusive. One of the simplest constitutive equations that is able to describe their main rheological features and has been abundantly used in recent research,\textsuperscript{17,21} is the model known as FENE-P\textsuperscript{23} (acronym for “finitely extensible nonlinear elastic” model with Peterlin’s closure\textsuperscript{3}). Further details are given in Sec. II.

With regard to the boundary layer flow of FENE-P fluids, Olagunju\textsuperscript{27,28} proposed an approximate similarity solution in contrast to the Newtonian case for which there is a global self-similar solution.\textsuperscript{29} Olagunju called it a local self-similar solution, because it still depended on two independent spatial variables, the streamwise coordinate and the Newtonian similarity variable, as will be discussed in Sec. IV. In his works, Olagunju only provided information on the profiles of velocity and on the law of variation for the friction coefficient. No information was given on other relevant quantities such as the laws of variation of the boundary layer, displacement, and momentum thicknesses, on the profiles of the polymer stress contribution, and on the effects of maximum polymer extensibility ($L^2$) and polymer concentration ($\beta_p$) (although his solution contained $L^2$ and $\beta_p$, he did not explore their effects). In addition, Olagunju ignored non-negligible elastic contributions to the rheological constitutive equation, which did not affect significantly the velocity profiles, but certainly do affect the polymer stresses as was found for the laminar planar jet flows of viscoelastic FENE-P fluids by Parvar et al.\textsuperscript{30} In particular, Parvar et al.\textsuperscript{30} showed that an Olagunju type of solution of the planar jet is unable to properly estimate the normal streamwise component of the polymer conformation tensor ($C_{xx}$) at very low Wi and also the normal transverse component ($C_{xy}$) in the whole range of Wi. Bearing in mind that the wall in a boundary layer flow imposes significantly higher rates of deformation, and consequently higher elastic stresses, than those existing in a planar jet (for similar characteristic fluid velocities), Olagunju’s polymer stress simplifications\textsuperscript{27} needs to be assessed, while also providing the missing information on boundary layer flow characteristics. This sets the stage for the present work in which we provide a different and more complete solution to the boundary layer flow over a flat plate at zero pressure gradient for FENE-P fluids.

Section II presents the flow set up, coordinate system, and full set of governing equations. The order of magnitude analysis leading to a simplified momentum equation is carried out in Sec. III. Section IV addresses the simplifications of the constitutive equation and introduces the mathematical transformations leading to the final coupled set of equations. This set of equations is numerically solved with numerical methods explained in Sec. V, and Sec. VI presents and discusses its results in detail, in some cases comparing them with the results of the full set of governing equations numerically solved with the RheoFoam toolbox of OpenFoam.

II. FLOW PROBLEM AND GOVERNING EQUATIONS

The laminar planar flat plate flow is sketched in Fig. 1. A uniform free stream velocity ($U_{\infty}$) flows over a thin immobile semi-infinite flat plate of length $L$ at zero incidence, and a streamwise null pressure gradient is imposed outside the boundary layer, whose thickness is denoted by $\delta$. The origin of the coordinate system is placed at the plate leading edge, with $x$, $y$ denoting the streamwise and transverse coordinates, respectively. The no-slip boundary condition is imposed at the wall, and the fluid velocity increases with the wall distance across the boundary layer approaching asymptotically the free-stream value. To define the boundary layer thickness, the criterion of local streamwise velocity equal to 99% of the free stream velocity is used.\textsuperscript{17,21,22}

The viscoelastic dilute polymer solutions are described by the finitely extensible nonlinear elastic model\textsuperscript{23} in which the Peterlin closure\textsuperscript{1} provides a closed form to compute the average dumbbell length needed to determine the restoring spring force of the kinetic-theory physical model dumbbell, the so-called FENE-P constitutive equation. This model has the minimum ingredients needed to describe the rheology of dilute polymer solutions, such as memory effects, shear-thinning behavior, and bounded elastic stresses. Hence, it has been a popular choice to investigate fundamental laminar as well as turbulent flow behavior of dilute polymer solutions (Refs. 9, 22, and 33 to name a few).

The governing equations for steady flow of incompressible fluids are written next in indicial notation. The conservation of mass is

$$\frac{\partial u_i}{\partial x_i} = 0$$

(1)

and the momentum equation is

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i}$$

(2)

where $u_i$ is the velocity vector, $P$ is the pressure, and $\rho$ is the fluid density. The fluid extra stress $\tau_{ij}$ is given as the sum

$$\tau_{ij} = \tau_{ij}^n + \tau_{ij}^p$$

(3)

of the Newtonian solvent stress ($\tau_{ij}^n$) and a polymer contribution ($\tau_{ij}^p$).

The solvent stress is

$$\tau_{ij}^n = 2 \mu_n S_{ij}$$

(4)

where $\mu_n$ is its kinematic viscosity and $S_{ij}$ is the rate-of-strain tensor defined by
The polymer stress contribution \( \tau_{ij}^{p} \) is given by the FENE-P model \(^{23,24} \) as
\[
\tau_{ij}^{p} = \frac{\nu_{p} \lambda}{\lambda} \left[ f(C_{kk}) C_{ij} - f(L) \delta_{ij} \right],
\]
where \( \nu_{p} \) is the zero-shear rate polymer kinematic viscosity coefficient, \( \lambda \) is the longest relaxation time of the polymer molecules, \( \delta_{ij} \) is the identity tensor, and \( C_{ij} \) is the dimensionless conformation tensor. This tensor expresses the orientation and stretch of the model polymer dumbbell at each flow point (cf. Ref. 23) and needs to be given by an adequate evolution equation. Finally, \( f(C_{kk}) \) is the Peterlin function, a scalar function of the trace of the conformation tensor and \( f(L) \) is usually its equilibrium value (value at rest) that depends on the square of the maximum normalized dumbbell extensibility, \( L^2 \).

There are several variants of the FENE-P model that differ in the Peterlin function used of which we single out the following three, which give essentially identical responses provided \( L^2 \gg 3 \):
\[
\begin{align*}
 f(C_{kk}) &= \frac{L^2}{L^2 - C_{kk}} \quad \text{and} \quad f(L) = \frac{L^2}{L^2 - 3}, \quad (7a) \\
 f(C_{kk}) &= \frac{L^2}{L^2 - C_{kk}} \quad \text{and} \quad f(L) = 1, \quad (7b) \\
 f(C_{kk}) &= \frac{L^2 - 3}{L^2 - C_{kk}} \quad \text{and} \quad f(L) = 1. \quad (7c)
\end{align*}
\]
The form of Eq. (7a) is the original, modified to Eq. (7b) by Bird et al. \(^{34} \) as discussed in the study by Beris and Edwards. \(^{33} \) The functions in Eq. (7c) have been used first by Vaithianathan and Collins \(^{35} \) and since then have been used extensively in investigations of turbulent flows of polymer solutions. \(^{21,22,36,37} \) Even though the formulation to be presented is general and independent of the model, the final numerical results pertain to the last set of functions.

In all cases, the evolution equation for the dimensionless conformation tensor is the following hyperbolic differential equation:
\[
\frac{\partial C_{ij}}{\partial t} + u_{k} \frac{\partial C_{ij}}{\partial x_{k}} = C_{ij} \frac{\partial u_{i}}{\partial x_{j}} + C_{ij} \frac{\partial u_{j}}{\partial x_{i}} - \frac{2}{\lambda} [f(C_{kk}) C_{ij} - f(L) \delta_{ij}],
\]
(8)

III. MOMENTUM EQUATION

We follow a standard procedure for boundary layer analysis (cf. Refs. 31 and 32): we start by making the governing equations dimensionless prior to simplifying them through an order of magnitude analysis. In this process, we adopt the analysis performed by Parvar et al. \(^{24} \) for the planar jet flow of FENE-P fluids. To normalize the functions, the length of the domain \( L \) and the boundary layer thickness \( \delta \) are used as characteristic streamwise and transverse length scales, whereas the free-stream velocity \( (U_{\infty}) \) is used as the characteristic velocity scale. The Reynolds number \( (Re_{L}) \) is defined by
\[
Re_{L} = \frac{U_{\infty} L}{\nu_{0}},
\]
where \( \nu_{0} \) is the zero-shear rate kinematic viscosity of the solution \( (\nu_{0} = \nu_{v} + \nu_{p}) \). We define the ratio between the polymer viscosity and the solution zero-shear rate viscosity, \( \beta_{p} \),
\[
\beta_{p} = \frac{\nu_{p}}{\nu_{0} + \nu_{p}} = \frac{\nu_{p}}{\nu_{0}},
\]
(10)

which is proportional to the polymer concentration. The Weissenberg number \( (Wi_{L}) \) is
\[
Wi_{L} = \frac{\lambda U_{\infty}}{L}.
\]
(11)
The following normalized lengths and velocities (denoted with \( \ast \)) are defined to ensure that their dimensionless derivatives in the continuity equation are at most of the order of unity (in particular \( L \gg \delta \))
\[
x_{\ast} = \frac{x}{L}, \quad y_{\ast} = \frac{y}{\delta}, \quad u_{\ast} = \frac{u}{U_{\infty}}, \quad v_{\ast} = \frac{vL}{U_{\infty} \delta}, \quad p_{\ast} = \frac{P}{\rho U_{\infty}^{2}}.
\]
(12)

Back-substitution into the continuity equation leads to
\[
\frac{\partial u_{\ast}}{\partial x_{\ast}} + \frac{\partial v_{\ast}}{\partial y_{\ast}} = 0.
\]
(13)
The normalized \( x \)-momentum equation becomes
\[
\frac{\partial u_{\ast}}{\partial x_{\ast}} + \nu_{\ast} \frac{\partial u_{\ast}}{\partial y_{\ast}} = -\frac{\partial p_{\ast}}{\partial x_{\ast}} + \frac{(1 - \beta_{p}) L^{2}}{Re_{L}} \left( \delta \frac{\partial^{2} u_{\ast}}{\partial y_{\ast}^{2}} + \frac{\partial^{2} u_{\ast}}{\partial y_{\ast} \partial x_{\ast}} \right) + \frac{\beta_{p}}{Wi_{L} Re_{L} \delta} \left( \frac{\delta \partial [f(C_{kk}) C_{xx} - 1]}{\partial x_{\ast}} + \frac{\partial [f(C_{kk}) C_{yy}]}{\partial y_{\ast}} \right),
\]
(14)
As for the planar jet, \(^{26} \) the following relations are valid:
\[
\begin{align*}
\nu_{\ast} &= \frac{(1 - \beta_{p}) L^{2}}{Re_{L}} \left( \delta \frac{\partial^{2} u_{\ast}}{\partial y_{\ast}^{2}} + \frac{\partial^{2} u_{\ast}}{\partial y_{\ast} \partial x_{\ast}} \right) \quad \text{and} \quad \beta_{p} \frac{\partial [f(C_{kk}) C_{yy}]}{\partial y_{\ast}} \rangle
\end{align*}
\]
\[
\gg \frac{\beta_{p} L}{Wi_{L} Re_{L} \delta} \left( \frac{\delta \partial [f(C_{kk}) C_{xx} - 1]}{\partial x_{\ast}} + \frac{\partial [f(C_{kk}) C_{yy}]}{\partial y_{\ast}} \right),
\]
(15)
which indicates the classical boundary layer proportionality, and
\[
\frac{\beta_{p}}{Wi_{L} Re_{L} \delta} \left( \frac{L}{\delta} \right) = \frac{1}{O(1)} \rightarrow Wi_{L} = O\left( \beta_{p} Re_{L}^{-1} \right).
\]
(16)
Equation (16) implies small values of \( Wi_{L} \), for which it can be shown from the constitutive equation that \( C_{yy} \approx \lambda C_{yy} \partial u_{\ast} / \partial y_{\ast} \). However, the simplified momentum equation can still be used at higher values of \( Wi_{L} \), when \( C_{yy} \) increases significantly, provided the shear stress gradient terms dominate over the neglected streamwise gradient of the polymer normal stress. This happens to be the case until very high values of \( Wi_{L} \), when the simplified constitutive equation is no longer valid, i.e., the simplifications in the constitutive equation breakdown before those of the momentum equation, as will be shown.

The order of magnitude analysis of the \( y \)-momentum equation leads to \( \partial p_{\ast} / \partial y_{\ast} = 0 \). Since the present analysis is directed to a flat plate boundary layer \( \partial p_{\ast} / \partial x_{\ast} = 0 \), the pressure is constant in the
entire flow domain and the y-momentum equation is no longer needed. We proceed with the dimensional form of the simplified x-momentum equation
\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu_a \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial f(C_{ik})}{\partial y} \right). \] (17)

Through the introduction of the stream function \( \psi \), continuity is enforced and only the x-momentum equation needs to be solved. The stream function \( \psi \) is defined as
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \] (18)

Back-substituting into the x-momentum equation leads to
\[ \left( \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial^2 \psi}{\partial x \partial y} \right) - \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right) = \nu_a \left( \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{\nu_p}{\lambda} \frac{\partial f(C_{ik}) C_{xy}}{\partial y}. \] (19)

To solve this equation, we need the variations of \( C_{ij} \) and \( f(C_{ik}) \) in order to obtain a closed form solution. The stream function and all components of the conformation tensor depend simultaneously on \( x \) and \( y \), but the use of the approximate similarity transformation discussed below will compact the solution.

IV. CONFORMATION TENSOR EQUATION

Under steady state conditions Eq. (8) simplifies to
\[ \frac{\partial C_{ij}}{\partial x_k} = C_{jk} \frac{\partial u}{\partial x} + C_{ik} \nu_a \frac{\partial u}{\partial x} - \frac{1}{\lambda} \left( f(C_{ik}) C_{ij} - f(L) \delta_{ij} \right). \] (20)

In flows with high shear rates, the distortion and stress contributions [terms on the right-hand-side of Eq. (20)] are usually the relevant quantities, as shown by the order of magnitude analysis of Parvar et al.\(^{30}\) for the planar jet. Substituting the velocities, as defined through the stream function in Eq. (19), the simplified evolution equations for the non-zero components of the conformation tensor are rewritten as
\[ -2 \left( C_{xx} \frac{\partial^2 \psi}{\partial x \partial y} + C_{xy} \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{1}{\lambda} \left( f(C_{ik}) C_{xx} - f(L) \right) = 0, \] (21)
\[ 2 \left( C_{xy} \frac{\partial^2 \psi}{\partial x^2} + C_{yy} \frac{\partial^2 \psi}{\partial x \partial y} \right) + \frac{1}{\lambda} \left( f(C_{ik}) C_{xy} - f(L) \right) = 0, \] (22)
\[ -\frac{1}{\lambda} \left( f(C_{ik}) C_{zz} - f(L) \right) = 0, \] (23)
\[ -C_{yy} \frac{\partial^2 \psi}{\partial y^2} + C_{zz} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\lambda} \left( f(C_{ik}) C_{yy} \right) = 0. \] (24)

These equations can be further simplified by considering \( \partial \psi/\partial y \gg \partial \psi/\partial x \) and that with the exception of very low Weissenberg number flows, the normal components of \( C_{ij} \) are higher than the shear components (at rest \( C_{xx} = C_{yy} = C_{zz} = 1 \) and \( C_{ij}(y/z) = 0 \)). Hence, we keep cross derivative terms when multiplying normal components of \( C_{ij} \), leading to
\[ -2 \lambda C_{xx} \frac{\partial^2 \psi}{\partial x \partial y} - 2 \lambda C_{xy} \frac{\partial^2 \psi}{\partial y^2} + f(C_{ik}) C_{xx} = f(L), \] (25)
\[ 2 \lambda C_{xy} \frac{\partial^2 \psi}{\partial x \partial y} + f(C_{ik}) C_{yy} = f(L), \] (26)
\[ f(C_{ik}) C_{zz} = f(L), \] (27)
\[ -\lambda C_{yy} \frac{\partial^2 \psi}{\partial y^2} + f(C_{ik}) C_{yy} = 0. \] (28)

The underlined terms in the equations for \( C_{xx} \) and \( C_{yy} \) are new, i.e., they are not present in Olgunju’s solution\(^{28,29}\) and through Peterlin’s function, they also affect the other components of \( C_{ij} \). However, since their effect on the polymer shear stress is weaker than on normal stresses, the impact on the velocity profiles will be weak as will be shown later.

Further manipulation of Eqs. (25)–(28) provides the following final expressions for the conformation tensor components
\[ C_{xx} = \frac{f(L) + 2 \lambda C_{oo} \frac{\partial^2 \psi}{\partial y^2}}{f(C_{ik}) - 2 \lambda \frac{\partial^2 \psi}{\partial x \partial y}}, \] (29)
\[ f(C_{ik}) \left( 2 \lambda \frac{\partial^2 \psi}{\partial x \partial y} + f(C_{ik}) \right) + 2 \lambda \frac{\partial^2 \psi}{\partial y^2} \] (30)
\[ C_{yy} = \frac{f(L)}{f(C_{ik})}, \] (31)
\[ C_{zz} = \frac{\lambda C_{yy} \frac{\partial^2 \psi}{\partial y^2}}{f(C_{ik}) - 2 \lambda \frac{\partial^2 \psi}{\partial x \partial y}}. \] (32)

In this set of coupled algebraic equations, \( C_{ij} \) depends on the flow characteristics, via the stream function, other \( C_{ij} \) components and its trace through the Peterlin function. The determination of the Peterlin function, defined in Eq. (7), is explained below. From the three normal components above, the trace \( C_{ik} \) is given by
\[ C_{kk} = \frac{3 f(C_{ik})^2 + 2 \lambda f(C_{ik}) \left( \frac{\partial^2 \psi}{\partial y^2} \right) - 4 \lambda \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2}{f(C_{ik}) \left( f(C_{ik})^2 - 4 \lambda \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right)}. \] (33)

Next, we introduce variables consistent with the self-similar Newtonian thin boundary layer solution\(^{31,32}\) and in dimensionless
form for generality. Here, the following variables \( \eta \) and function \( G(\eta, x) \) are utilized:

\[
\eta = \sqrt{\frac{U_\infty y}{2 \nu_0 x^{1/2}}}, \quad G(\eta, x) = \frac{\psi}{\sqrt{2U_\infty \nu_0 x^{1/2}}}. \tag{34}
\]

The streamwise and normal velocities are recovered from their definitions, as

\[
u = \frac{\nu_0 U_\infty}{2x} (\eta G'(\eta, x) - G(\eta, x)), \tag{35}
\]

\[
v = \frac{\nu_0 U_\infty}{2x} (\eta G''(\eta, x) - G(\eta, x)) \tag{36}
\]

where the prime indicates derivative in order to \( \eta \). For the corresponding Newtonian flow, these variables allow for a self-similar solution, and function \( G(\eta) \) only depends on \( \eta \). This is not the case for the FENE-P fluid, even if other powers of \( x \) and \( y \) are tried, obtaining a finding that Oalagonju\(^{[17,24]} \) had previously arrived at and, as investigated by Parvar\ et al.\(^{[17]} \) for the planar jet. Therefore, the flow problem remains two-dimensional, with \( G(\eta, x) \) depending on both \( \eta \) and \( x \), and this variable transformation does not simplify its solution unless further assumptions are introduced. This we do here by considering that streamwise variations of \( G(\eta, x) \) are negligible, i.e., henceforth \( \partial G(\eta, x)/\partial x \approx 0 \), allowing us to obtain what is here called an approximate similarity solution. As we shall see, this assumption still leads to a very good description of the velocity field, while allowing for a much simpler solution.\(^{[25-41]} \) Without it we would end up with a far more complicated set of equations bringing advantage over the numerical solution of the full set of original governing equations.

\[
\frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} \right) - \left( \frac{\partial \psi}{\partial x} \right)^2 = \nu_1 \frac{\partial^3 \psi}{\partial y^3} + \nu_1 f(L) \left( 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi}{\partial y^2} + f(C_{\psi}) \frac{\partial^4 \psi}{\partial y^4} - 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial f(C_{\psi})}{\partial y} \frac{\partial^2 \psi}{\partial y^2} \right) \tag{40}
\]

Back-substitution of \( K' \) into Eq. (41), and further mathematical manipulation, leads to the following final form of the momentum equation cast in terms of \( G(\eta) \):

\[
G'' = - \frac{G'/G'' - \beta_1 f(L)}{(1 - \beta_1) G'' + \beta_2 f(L) G''} \left( \frac{C_4 G''^2 + K G'' - K' G''}{(K - C_4 G''^2)} \right). \tag{41}
\]

with

\[
C_4 = \lambda U_\infty x^{-1} = \text{W}_{10} . \tag{42}
\]

Again, the dependence on \( x \) and \( \eta \) is clear, and this equation also involves the first derivative of \( K \) relative to \( \eta \), which is obtained from Eq. (38) and is given by

\[
K' = - \left( \frac{2C_3 (C_1 K + C_3) + 2G'' G'' (C_2 + \eta^2 (C_1 K + C_3))}{C_6} \right) \tag{43}
\]

with

\[
C_5 = \eta G''^2, \quad C_6 = (3K'' + 2C_2 K + C_1 (\eta G'')^2). \tag{44}
\]

The Peterlin function \( f(C_{\psi}) \) also depends on \( \eta \) and \( x \), as shown below, and for mathematical clarity henceforth we use

\[
K(\eta, x) = f(C_{\psi}). \tag{37}
\]

but note that it is not necessary to invoke the above simplifying assumption for this quantity, i.e., \( \partial K(\eta, x)/\partial x \neq 0 \).

Substitution of all expressions into Eq. (7) leads to the following third order algebraic equation for \( K(\eta, x) \):

\[
K^3 + C_4 K^2 + C_1 (\eta G''^2) K + C_2 G''^2 + C_3 (\eta G'')^2 = 0 \tag{38}
\]

with dimensionless coefficients

\[
C_0 = \frac{(3I - 3f(L) - L^2)}{L^2}, \quad C_1 = -2J U_\infty^2 x^{-2} = -W_{10}^2, \quad C_2 = -2J f(L) U_\infty^3 x^{-1} = -f(L) R_e W_{10}^2, \quad C_3 = \frac{2J^2 L^2 + f(L) - 3I U_\infty^2 x^{-2}}{L^2} = \frac{(L^2 + f(L) - 3I) W_{10}^2}{L^2}. \tag{39}
\]

Here, \( I = 1 \) if the Peterlin functions are given by Eq. (7c) and \( I = 0 \) otherwise. Coefficients \( C_1 \) to \( C_3 \) in Eq. (39) depend on \( x \); hence this shows why \( K \) depends on both \( \eta \) and \( x \), and for this single reason, it is not possible to get a full self-similar solution as for Newtonian fluids.

Regarding the \( x \)-momentum equation, by substituting \( C_{\psi \psi} \) [from Eq. (32)] into Eq. (17),

\[
G'' = - \frac{G'/G'' - \beta_1 f(L)}{(1 - \beta_1) G'' + \beta_2 f(L) G''} \left( \frac{C_4 G''^2 + K G'' - K' G''}{(K - C_4 G''^2)} \right). \tag{41}
\]
\[ \partial G(\eta, x)/\partial x \approx 0 \], each set is composed of an ordinary third-order differential equation on \( G(\eta) \) and an algebraic cubic equation for \( K(\eta, x) \).

To solve each set of equations, the third-order differential equation is converted to a system of ordinary first-order differential equations with the following transformations, \( G_1 = \frac{dG}{d\eta}, G_2 = \frac{dG_1}{d\eta}, \) and \( G_3 = G \), or alternatively written as

\[ \frac{dG_1}{d\eta} = G''', \]
\[ \frac{dG_2}{d\eta} = G_1, \]
\[ \frac{dG_3}{d\eta} = G_2. \]

The cubic equation is solved first with the Cardan–Tartaglia formula.44,45 Its correct solution must be real-valued, all the normal components of \( C_i \) are positive, and \( 3 \leq C_{ik} \leq L^2 \). The ensuing system of differential equations is solved numerically by a fourth-order Runge–Kutta method coupled with a shooting technique to match the boundary conditions for the laminar flat plate boundary layer flow are31,32

\[ G(\infty) \rightarrow 1, \quad G(0) = G'(0) = 0. \]  

VI. RESULTS AND DISCUSSION

A. Newtonian fluids: Validation

The governing equations for the boundary layer flow of the viscoelastic fluid reduce to those for a Newtonian flow for \( \beta_p = 0 \), and this is used to verify the solution.31 Through asymptotic analysis, White31 verified that at \( \eta = 10 \) the flow characteristics are very close to boundary conditions at infinity; therefore, the same condition is used here. We define the local Reynolds \( Re_x \) as per Eq. (9) with \( x \) instead of \( L \).

We considered a maximum \( Re_x = 1 \times 10^5 \), a flow condition for which the Blasius solution remains valid31 and below the critical condition for laminar-turbulent transition \( (Re_x,cr = 5 \times 10^5) \) for Newtonian fluids.

Figure 2(a) shows an excellent agreement between the numerical values of \( G, G' \), and \( G'' \) for the current solution and the literature. The plotted quantities are related to the velocity profiles \( u \) and \( v \) according to Eqs. (35) and (36). By considering Eqs. (34) and (35), the following characteristics of the boundary layer flow were also quantified:

- The boundary layer thickness (\( \delta \)) based on the 99% of free stream velocity criterion (cf. Sec. II) is quantified by setting \( G' = 0.99 \), giving \( \eta = 3.4723 \), and consequently
  \[ \delta = 4.9105 \frac{x}{\sqrt{Re_x}}. \]  

- The displacement thickness (\( \delta' \)) is defined as
  \[ \delta' = \int_0^\infty \left( 1 - \frac{u}{U_{\infty}} \right) dy = \frac{2\nu x}{U_{\infty}} \int_0^\infty (1 - G') d\eta = x \sqrt{\frac{2}{Re_x}} \lim_{\eta \to \infty} [\eta - G(\eta)], \]  
leading to \( \delta'/x = 1.7208/\sqrt{Re_x} \).

- The momentum thickness (\( \theta \)) is defined as
  \[ \theta = \int_0^{\infty} \frac{u}{U_{\infty}} (1 - \frac{u}{U_{\infty}}) dy = \frac{2\nu x}{U_{\infty}} \int_0^\infty G'(1 - G') d\eta = x \sqrt{\frac{2}{Re_x}} \int_0^\infty G'(1 - G') d\eta, \]  
leading to \( \theta/x = 0.664/\sqrt{Re_x} \), and a shape factor

\[ H = \delta'/\theta = 2.591, \]  
consistent with Refs. 31 and 48.
The local skin-friction coefficient $C_f$ for the Newtonian fluid is given by

$$C_f = \frac{\tau_{yy}}{\rho U_\infty^2} = \frac{\partial^2 \psi}{\partial y^2}.$$  (53)

The variations with $Re_x$ of these four quantities are plotted in Fig. 2(b), and the agreement is excellent with data from Refs. 31 and 32.

B. FENE-P fluids: Validation

We start with a comparison between our approximate similarity solution and the numerical solution of the full set of non-simplified governing equations using the RheoFoam toolbox of OpenFoam,\textsuperscript{25,26} and also with the approximate similarity solution of Olagunju.\textsuperscript{27} Olagunju\textsuperscript{27,28} only presented the velocity profiles and the law of variation for the friction coefficient, so we used his equations to extract other quantities.

The RheoFoam simulation relied on the use of the high-order resolution scheme CUBISTA\textsuperscript{49} for the advective terms in the momentum and conformation equations. The computational domain had a length $1.2L$ divided into two blocks: block I upstream the plate leading edge was $0.2L$, long and block II along the plate had a length of $L$. The width of both blocks was set equal to $2L$. Within each block, the non-uniform computational grid had $N_x \times N_y \times N_z$ cells for the $x$, $y$, $z$ directions, respectively, as given in Table I together with the expansion/contraction mesh factors $f_x = \Delta x^{-1}/\Delta x$ and $f_y = \Delta y^{-1}/\Delta y$, and ratios of mesh size over boundary layer thickness at some locations. This mesh was selected after an assessment of mesh independence using four grids. Differences between the results of this grid and those obtained in a grid with twice the number of cells in each direction are below 0.05%.

On the inlet boundary, a uniform velocity was imposed, on the outlet boundary, a zero gradient condition was set for all quantities, and no slip was imposed at the wall. At the boundary upstream the wall, in block I, symmetry conditions were set. At the boundary opposite the wall, far from the boundary layer, free stream velocity conditions were imposed, both in blocks I and II.

The simulation was carried out for the Peterlin function of Eq. (7c) with $\beta_p = 0.1$ and $L^2 = 900$ and the results shown pertain to $x/L = 0.2$, where the local Reynolds and Weissenberg numbers are $Re_x = 2 \times 10^4$ and $Wi_x = 0.1$, respectively [the local Weissenberg number $Wi_x$ is defined as per Eq. (11) with $x$ instead of $L$].

Figure 3 compares the transverse profiles of normalized streamwise and wall-normal velocities obtained in RheoFoam and with the present and Olagunju’s semi-analytical solutions showing excellent agreement, as shown through the zooms in Fig. 3. Olagunju’s solution very slightly over-predicts $u/U_\infty$ and $(v/U_\infty)\sqrt{Re_x}$.

The corresponding transverse profiles of the conformation tensor components ($C_{yy}$, $C_{xx}$, $C_{zz}$, and $C_{xy}$) are plotted in Fig. 4, and again there is a very good agreement between the results of the present work and of the RheoFoam simulation. However, Olagunju’s solution shows various differences relative to our solution and RheoFoam predictions: it over-predicts $C_{yy}$ near the wall and, regardless of $Wi_x$, it always predicts the maximum $C_{yy}$ to occur at the wall, whereas both the current solution and RheoFoam show that the maximum $C_{yy}$ occurs at a slight distance from wall, with this offset increasing with $Wi_x$.

Simultaneously, Olagunju’s solution over-predicts the wall peak of $C_{xx}$ in Fig. 4(d), whereas with regard to $C_{yy}$ and $C_{zz}$, Figs. 4(b) and 4(c) show that they are qualitatively incorrect even if the numerical values are not too different from 1, as they should be. As a matter of fact, the peak value of $C_{yy}$ away from the wall and the local minimum positive value at the wall are not captured qualitatively by Olagunju’s solution, which always predicts a wall peak value, whereas for $C_{zz}$

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
Block & $N_x$ & $N_y$ & $N_z$ & $f_x$ & $f_y$ & $\Delta x_{\text{cell}}/\Delta x_{\text{BL}}$ & $\Delta y_{\text{cell}}/\Delta y_{\text{BL}}$ \\
\hline
I & 20 & 600 & 1 & 0.886 & 1.0088 & $\ldots$ & $\ldots$
II & 200 & 600 & 1 & 1.011 & 1.0088 & 0.0439 & 0.003 & 0.439
\hline
\end{tabular}
\caption{Characteristics of the meshes used in the RheoFoam calculations for validation. The values of $\Delta x$ and $\Delta y$ are at the cells nearest the wall.}
\end{table}
FIG. 4. Comparison between transverse profiles of the conformation tensor components obtained with RheoFoam and present and Olagunju solutions at $\xi = 0.4$ for $\beta_p = 0.1$, $L^2 = 900$, $Re_x = 2 \times 10^5$, and $W_{\text{ix}} = 0.1$: (a) $C_{xy}$, (b) $C_{yy}$, (c) $C_{zz}$, (d) $C_{xx}$. Lines are a guide to the eye.

FIG. 5. Transverse profiles of normalized streamwise velocity ($u/U_\infty$): (a) from present solution at various flow conditions (dashed lines are a guide to the eye); (b) comparison between present solution (dashed dotted lines) and RheoFoam solution (solid lines) for $\beta_p = 0.1$, $L^2 = 900$, and $Re_x = 2 \times 10^5$. 

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plotted in Fig. 4(c), Olagunju’s values are in excess of 1, whereas the correct values are below 1.

As stated earlier, the discrepancies between both solutions come from the simplifications in the conformation equations for $C_{xx}$ and $C_{yy}$, then carried to the other components of $C_{ij}$ via the Peterlin function $f(C_{kk})$, and increase substantially with $W_i$. In contrast, at lower $W_i$, such as $W_i = 0.005$, velocity and conformation tensor profiles of both semi-analytical solutions collapse, but they are not shown for conciseness.

It is worth comparing the computational cost of the semi-analytical solutions and of the RheoFoam simulation. The latter was performed by a computer equipped with an Intel Xeon E5 processor with 12 MB L3 cache and Turbo Boost up to 3.9 GHz, with parallel processing using its six computer cores. The computational time was 4.5 h; however for the semi-analytical solutions, the same computer took about 1 s.

### C. FENE-P fluids: Results

The solution presented and discussed here pertains to the Peterlin function of Eq. (7c), and we vary the Reynolds and Weissenberg numbers as well as $\beta_p$ and $L^2$. The local Reynolds $R_e$...
and Weissenberg numbers $Wi_x$ and their values at the end of the plate ($Re_C$ and $Wi_C$) are both used. Note that a specific boundary layer flow sets $Re_L$ and $Wi_L$, the data pertain to different local values of $Re_x$ and $Wi_x$, since $Re_x \sim x$ and $Wi_x \sim x^{-1}$. Alternatively, one may compare physical quantities at the same local values of $Re_x$ and $Wi_x$, but then, at each $x/L$, data correspond to different values of $Re_L$ and $Wi_L$.

To show that the approximate similarity solution remains valid well beyond the small $Wi$ suggested by Eq. (16) [e.g., $Wi_L = O(10^{-3})$ for $Re_L = O(10^3)$] we include comparisons with data from RheoFoam simulations at $Re_L = 1 \times 10^5$, with $\beta_p = 0.1$, $L^2 = 900$, and $Wi_L = 0.2$ and $Wi_L = 1.0$. These simulations were carried out in a rectangular computational domain similar to that described in Sec. VI B and Table I. We faced numerical instabilities with the RheoFoam simulations at higher values of $Wi_L$.

1. Flow field

Figures 5(a) and 6(a) show transverse profiles of normalized streamwise and wall-normal velocities as a function of the independent dimensionless numbers. By normalizing lengths with the boundary-layer thickness and using the local Reynolds number ($Re_x$) for the wall-normal velocity, the profiles at low elasticity levels tend to collapse on the Newtonian profiles, but by increasing elasticity the profiles progressively deviate from the Newtonian asymptote thus showing that their similarity nature is only approximate. The deviation is small for the streamwise velocity profile, which is pinned at the wall and at the edge of the boundary layer in this normalization. For the transverse velocity, in general, there is a progressive decrease with elasticity as the boundary layer edge is approached, so that the dimensionless wall-normal velocities can be 10% smaller than the Newtonian values under certain conditions of high elasticity (high $Wi_x$ for non-dilute solutions, i.e., for high $\beta_p$).

Figures 5(b) and 6(b) compare transverse profiles of normalized velocities at $x/L = 0.2$. Even at $Wi$ numbers several orders of magnitude larger than imposed by Eq. (16) [note that $Wi_x = (L/L)(Wi_L)$], the present solution can still predict well the profiles of streamwise and wall-normal velocities. For the same global flow condition the comparison always improves on going downstream, because $Wi_x$ decreases (such profiles are not shown for conciseness).

The local skin-friction coefficient $C_f$ for the FENE-P fluid is given by
The streamwise variations of the friction factor \( \frac{\tau_{xy}}{2 \rho U_\infty^2} \) and of the normalized boundary layer thickness \( \frac{\delta}{x} \) are plotted in Figs. 7(a) and 8(a), respectively. As for the laminar planar jet previously evaluated, the present solution agrees well with RheoFoam results for Re\( x \) = 10\(^5\).
investigated, when elasticity effects are weak (low $W_i$, low $\beta_p$, high $L$) both quantities follow closely the corresponding Newtonian results, but as $W_i$ and $\beta_p$ increase or $L$ decreases, the boundary layer thickness and the skin friction coefficient decrease on account of the stronger shear-thinning nature of the shear viscosity accompanying those changes of the independent dimensionless numbers. The comparison between the approximate similarity semi-analytical solution and the RheoFoam predictions for the friction factor ($C_f$) in Fig. 7(b) and for the normalized boundary layer thickness ($\delta/x$) in Fig. 8(b) are very good at $W_i$ well in excess of the limit imposed by Eq. (16), with differences below 2% for $(\delta/x)$ and 1% for $C_f$. In these plots and in Figs. 10(b) and 11(b), since $W_{ix} = W_{iz}$ at $x$ where $Re_z = 10^5$, then $W_{ix} = 5W_{iz}$ where $Re_z = 10^5$.

Before proceeding, the critical issue of the characteristic length used to define the Weissenberg number needs to be discussed, because it has implications on the interpretation of what are low or high Weissenberg number flows. This issue has similarities with the definition of the Reynolds number in classical boundary layer theory. We use a distance measured from the plate leading edge (the local distance, $x$, or the full length of the plate, $L$) as this makes the definition of $W_i$ (and of $Re$) independent of the solution. However, from a physical point of view it makes more sense to use the boundary layer thickness in the ratio of time scales, since the flow is dominated by shear (except at very large $W_i$ where extensional effects may become important as explained above). The problem is that $\delta$ is part of the solution and it is always difficult to know a priori what the value of $W_{ix}$ (and $Re_z$) will be. For instance, for the cases in Figs. 7 and 8 where $0.01 < W_{ix} < 5.0$, one gets $0.3 < W_{ix} < 150$ as shown in Fig. 9, i.e., the use of the more physically meaningful boundary layer thickness $\delta$ as characteristic length leads to larger numerical values of $W_{ix}$, well in excess of $O(1)$. Therefore, the highest values of $W_{ix}$ used here to investigate the flow characteristics of the approximate similarity solution are large enough to assess the effects of strong elasticity, provided the solution is within its range of validity, an issue discussed later.

![Image](https://example.com/image.png)

**FIG. 12.** Streamwise variation of local peak values of the polymer stress and total shear stress for various flow conditions: (a) $\tau_{xx}$, (b) $\tau_{yy}$, (c) $\tau_{xy}$, and (d) $\tau_{xy}$. Data are normalized by the corresponding stress at $x/L = 0.2$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.
Figures 10 and 11 show the corresponding streamwise variations of the dimensionless displacement and momentum thicknesses. At low elasticity levels both quantities follow closely the corresponding Newtonian laws, however by increasing the elasticity level $\delta^*/x$ and $\theta/x$ decrease on account of the fuller velocity profiles associated with the shear-thinning of the shear viscosity. The comparisons with RheoFoam predictions in parts (b) show again very good agreement, with differences not exceeding 1% at the highest $Wi$ numbers plotted.

### 2. Conformation and stress tensors

Figures 12(a)–12(d) follow the streamwise decrease of the peak values of the polymer stress components and of the total shear stress, normalized by the corresponding peak values at $x/L = 0.2$. For viscoelastic fluids, the local Weissenberg number increases to infinity as the leading edge is approached; therefore, the boundary layer theory ceases to be valid between the leading edge, where its assumptions do not hold even for Newtonian fluids, and some location downstream. Hence, only data for $x/L \geq 0.2$ are shown.

**FIG. 13.** Streamwise variation of the ratio of wall polymer shear stress ($\tau_{yy}$) to wall solvent shear stress ($\tau_{yy}$). Lines are a guide to the eye; the solid line is the low elasticity asymptote.

**FIG. 14.** Streamwise variation of ratios between local maximum polymer ($\tau_{yy}$) over local maximum solvent stresses ($\tau_{yy}$): (a) $\frac{\tau_{yy}^{\text{max}}}{\tau_{yy}^{\text{max}}} \quad \text{(b)} \frac{\tau_{xx}^{\text{max}}}{\tau_{xx}^{\text{max}}}$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.

**FIG. 15.** Variation across the boundary layer of $\tau_{yy}^{\text{w}/\tau_{yy}^{\text{w}}}$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.
For all components, the normalized data plotted in Fig. 12 follow equations of the form

$$\log_{10}\left( \frac{s_{ij}^{\text{max}}}{C_{ij}} \right) = m\log_{10}\left( \frac{x}{L} \right) - b, \quad (56)$$

which are also shown as lines. At low elasticity levels, the normalized peak stress curves tend to an asymptote, which is indicated in each plot as a solid line.

The decay rate of the low elasticity asymptotic for $s_{xx}$ is twice as fast as for the shear stresses, because in this limit $s_{xx} \propto (\partial u/\partial y)^2$, whereas the polymer shear stress varies linearly with $\partial u/\partial y$ as does the Newtonian solvent shear stress, and consequently the total shear stress $\tau_{xy}$. The transverse normal stress $s_{yy}$ is much smaller than $s_{xx}$ and the rate of decay of its local peak depends on the streamwise variations of $\partial v/\partial y$ and $C_{yy}$. Further manipulation of Eqs. (34)–(36) shows that the streamwise variations of $\partial v/\partial y$ and $\partial u/\partial y$ are related, but the quantities involved are very small and simple order of magnitude arguments are unable to explain the rate of decay shown in Fig. 12(b) that only accurate numerical calculations can provide.

For low elasticity flows, the linear dependence of both shear stresses on $\partial u/\partial y$ is equivalent to having a constant ratio for $s_{xy}/s_{yy}$ equal to the ratio of viscosity coefficients $\nu_p/\nu_s = 1/9$ for $\beta_p = 0.1$ and $\nu_p/\nu_s = 1$ for $\beta_p = 0.5$, as shown in Fig. 13. On account of shear-thinning effects, as $Wi$ increases there is a decrease in the stress ratio $s_{xy}/s_{yy}$ and in the decay rates for all stresses shown.

The ratios of local maximum polymer stress over the corresponding local maximum solvent stress ($s_{ij}^{\text{max}}/s_{ij}^{\text{max}}$) show similar behaviors for the shear and wall-normal components. At low elasticity levels, the maximum shear stresses are at the wall so the behavior is that of Fig. 14, but as elasticity increases, the peak polymer shear stress moves away from the wall to the near wall vicinity, where the corresponding conformation tensor component peaks, and the ratio

**FIG. 16.** Transverse profiles of conformation tensor component quantities normalized by their corresponding absolute local peak values for various flow conditions: (a) $C_{xy}$, (b) $C_{yy}/C_0$, (c) $C_{xx}$, and (d) $C_{yy}$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.
The ratio $\frac{\tau_{yy}}{\tau_{y\infty}}$ also depends on the ratio $\frac{\nu}{\nu_s}$, with $\tau_{yy}$ and $\tau_{y\infty}$ their maxima being offset from the wall, where under low shear rate conditions they essentially depend linearly on $\partial u/\partial y$; hence, their ratio is also equal to the ratio of viscosity coefficients $\nu/\nu_s$, cf. Fig. 14(a). However, $\tau_{yy}$ and $\tau_{y\infty}$ behave differently as depicted in Fig. 14(b): whereas $\tau_{yy}$ is purely viscous and depends only on the small value of $\partial u/\partial x$, being negative, the polymer stress $\tau_{y\infty}$ exhibits the nearly quadratic dependence on the large values of $\partial u/\partial y$ and is positive; therefore, the ratio $\frac{\tau_{yy} \max}{\tau_{y\infty} \max}$ takes on large values even in the limit of small elasticity and increases with $W_i$ or $\beta_p$ and in inverse proportion to $L^2$.

Figure 15 plots the variation across the boundary layer of $\frac{\tau_{yy}}{\tau_{y\infty}}$ at some representative locations, as given through the values of $Re_s$ and $Wi_s$, showing that there are small variations significant and satisfying that the peak polymer stress values may occur at some distance from the wall. As the edge of the boundary layer is approached and the shear rates decrease, this stress ratio approaches the viscosity coefficient ratio $\nu/\nu_s = \beta_p/\left(1 - \beta_p\right)$. Note however that the variation of the total shear stress across the boundary layer is monotonic, with peak values at the wall (such profiles are not shown for conciseness).

At this stage, it is important to remember that the approximate similarity nature of the solution is introduced by the constitutive equations. Transverse profiles of conformation tensor component quantities, normalized by the corresponding local absolute peak values, are plotted in Figs. 16(a)–16(d) for a wide range of conditions at two different locations (expressed through the values of $Re_x$ and $Wi_x$). As expected, at low elasticity flow conditions, here represented by $Wi_s \leq 0.005$, the profiles collapse on asymptotic curves (solid lines); for increasing levels of viscoelasticity the profiles progressively deviate from the asymptotes and the nature of the approximate similarity behavior sets in. The normalized profiles of $C_{yy}$, $C_{x\infty} - 1$ and $C_{y\infty} - 1$ all exhibit their large variations taking place further away from the wall than in their asymptotic curves, and for the shear component, the location of the peak value also moves away from the wall. $C_{yy}$ is always positive, but the large variations of the normalized profiles of $C_{yy} - 1$ are a consequence of the ratio involving very small numbers due to very small variations of $C_{yy}$ around 1. At the wall $C_{yy} < 1$, hence the negative values are plotted. In addition, whereas at low elasticity levels the maximum $C_{yy} - 1 > 0$ and occurs away from the wall, at high elasticity levels, the maximum difference $C_{yy} - 1 < 0$ and occurs at the wall; therefore the normalized profiles are equal $+1$ and $-1$ at those specific locations, respectively. When viscoelasticity is not weak, Fig. 16 shows some degree of variation in terms of the normalized conformation tensor quantities due again to the amplification of small values in the ratio and also to the approximate nature of the similarity solution.

Next, Fig. 17 plots transverse profiles of $\tau_{yy}$ and $\tau_{y\infty} = \tau_{yy} + \tau_{y\infty}$, normalized by the corresponding local peak values, to show better the asymptotic behavior under conditions of weak viscoelasticity. Indeed, the plot of the total shear stress seems to be universal, but the zoom at the inset shows the existence of small differences. These differences are small because the approximate nature of the viscoelastic solution comes through the constitutive equation, i.e., through the polymer stress not through the momentum equation or the solvent stress. Both shear stresses are essentially determined by the magnitude of the shear rate $\partial u/\partial y$, and the solvent stress partially compensates for the variations of the polymer shear stress, so that the total stress profile has a near-universal shape. In Fig. 5, we also observed that the profile of normalized streamwise velocity is very weakly dependent on flow conditions showing a near-universal behavior.

Contrasting with the large variations in normalized $C_{yy} - 1$ shown in Fig. 16(d), the corresponding normalized $\frac{\tau_{yy}}{\tau_{yy} \max}$ in Fig. 17(a) presents a significantly better behavior because the small variations of $\left(C_{yy} - 1\right)$ are partially compensated by the small variations in $f(C_{y\infty})$ leading to a better behaved stress [note that $\tau_{yy} = f(C_{y\infty})(C_{yy} - 1)$].

For all flow conditions investigated, the peak values of $\tau_{yy}$ and of the total shear stress $\tau_{y\infty}$ are at the wall. For the polymer shear stress,
under weak elasticity conditions the peak value occurs at the wall and then progressively deviates away from the wall as elasticity and shear-thinning further increases. The location of the peak values and its variation as a function of flow conditions are plotted in Figs. 18(a) and 18(b) for $\tau_{xy}$ and $\tau_{yy}$, respectively, including the equations for the asymptotic locations.

The streamwise variation of the polymer wall shear stress ($\tau_{xy}$) is plotted in Fig. 19, confirming the findings from the transverse profiles. The low elasticity asymptote is also plotted for polymer shear stress.

To assess the limit of validity of the approximate solution in terms of conformation tensor components, transverse profiles of $C_{yy}$, $C_{xy}$, $C_{zz}$ and $C_{xx}$ are plotted and compared in Fig. 20 with the corresponding RheoFoam results, at two different locations ($x/L = 0.2$ and 1) for two different flow conditions ($Wi_L = 0.2$ and 1), in order to encompass a wide range of $Wi_L$. Conformation tensor components are the most sensitive quantities to flow elasticity, and discrepancies between the simplified theory and predicted RheoFoam simulations are now clear for the highest $Wi_L$ profiles. The differences are related to large peak values in $C_{xx}$, $C_{xy}$ and $C_{yy}$ appearing off the wall, at $0 \leq y/\delta \leq 1$, with the transverse RheoFoam profiles exhibiting changes in concavity. The corresponding $C_{zz}$ do not exhibit local maxima, but very rapid variations the simplified theory is unable to follow.

The “well-behaved” semi-analytical solution is unable to follow these large variations in $C_{ij}$ at high $Wi_L$ flows, on account of the imposed boundary layer simplifications, but the values at the wall are always correctly described. Decreasing $Wi_L$ by going downstream reduces the differences, and at $Wi_L = 0.2$, the behavior exhibited by the RheoFoam simulation is now well described by the semi-analytical solution, whereas at $Wi_L = 0.3$, the RheoFoam profiles show the appearance of local peaks, i.e., the critical value of $Wi_L$ is here between $0.2$ and $0.3$. Note that at $Wi_L = 0.2$ and for $L^2 = 900$, the extension of the dumbbells, measured by $C_{kk}/L^2$, is around $0.3$, hardly characteristic of low elasticity flows. In spite of the poor performance of the semi-analytical solution in describing profiles of $C_{ij}$ at $Wi_L > 0.2$, the semi-analytical solution remains accurate in terms of friction coefficient, normalized velocity profiles, and normalized momentum, displacement, and boundary layer thicknesses up to $Wi_L \approx 5$ as seen in Sec. VII C.1.

For $Re_L = 0.5$, with $L^2 = 900$, and increasing $L^2$ from $900$ to $10000$ for $\beta_p = 0.1$, maintains the critical value of $Wi_L$ between $0.2$ and $0.3$, but the corresponding profiles are not shown for conciseness. Reducing $L^2$ from $900$ to $400$, for the same $\beta_p$, slightly reduces the critical $Wi_L$ to $0.2$ (local peaks are seen to start appearing only for some components of $C_{ij}$).

FIG. 18. Location of peak values of some stress components: (a) Normalized $y$ for $\tau_{xy}$, and (b) normalized $y$ for $\tau_{yy}$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.

FIG. 19. Decay law for the polymer wall shear stress ($\tau_{xy}$) normalized by its value at $x/L = 0.2$. Lines are a guide to the eye; the solid line is the low elasticity asymptote.
VII. CONCLUSIONS

The boundary layer flow of FENE-P fluids over a flat plate at zero pressure gradient, initially investigated by Olagunju, was revisited leading to a more accurate simplified set of conformation tensor equations that still allowed a semi-analytical solution to be obtained. Due to the fluid viscoelasticity this solution showed an approximate self-similar behavior, contrasting with the full similar behavior of the corresponding Newtonian problem.

The current semi-analytical solution is more accurate than Olagunju’s solution, and in addition, we provide a more extensive set of results of the flow characteristics that include profiles of polymer stress and conformation tensor components, asymptotic laws of decay of peak stresses and conformation tensor components in the limit of low elasticity as well as laws for the location of such peak values. We also present results quantifying the boundary layer thickness as well as the displacement and momentum thicknesses. Even though the peak polymer shear stresses take place away from the wall, except in the low elasticity limit where the polymer and solvent shear stresses are proportional to each other, the peak of the total fluid shear stress is always at the wall.

For large values of $Wi_k$ and $\beta_p$, combined with low values of $L^2$, one of the assumptions of the boundary layer theory starts to break down. As is well known, the boundary layer theory is not valid in the vicinity of the plate leading edge, even for a Newtonian flow, but since $Wi_k \propto x^{-1}$, this region extends further downstream of the leading edge for viscoelastic fluids. In order to assess the shortcomings of the semi-analytical solution under these conditions, and in particular at large and very large Weissenberg numbers, we provide results from the numerical solution of the full set of complete (non-simplified) governing equations. For local Weissenberg numbers ($Wi_k$) of up to about 0.2 (for $\beta_p = 0.1$ and $L^2 = 900$), the semi-analytical solution approximates reasonably well all flow characteristics, but as $Wi_k$ further increases the transverse variation of the conformation tensor components become increasingly complex within $0.3 \leq y/\delta \leq 1$, exhibiting changes in the concavity of the profiles that go together with the development of large peak stress values. Nevertheless, the semi-analytical solution continues to predict well the wall values of all components of the stress and conformation tensors. In addition, and in spite of its poor performance in predicting $C_0$ for $Wi_k > 0.2$, it still predicts the normalized velocity profiles, friction coefficient and normalized evolution of the momentum, displacement and boundary layer thicknesses within $5\%$ of the RheoFoam solutions for $Wi_k \leq 5$. For $\beta_p = 0.5$, with $L^2 = 900$, and increasing $L^2$ from 900 to 10,000 for $\beta_p = 0.1$, maintained the critical value of $Wi_k$ for the conformation tensor between 0.2 and 0.3. Reducing $L^2$ from 900 to 400, for the same $\beta_p$, slightly reduces the critical $Wi_k$ to 0.2.

The different forms of the FENE-P model in Eq. (7) only differ in behavior when the extent of the dumbbells, as measured by $C_{kk}$, becomes significant in comparison with $L^2$. Therefore, the current semi-analytical solution should not depend on the form of the FENE-P model except if close to its limit validity condition, as quantified by the critical value of the local Weissenberg number, the ratio $C_{kk_{\text{max}}}/L^2$ approaches 1. We did not investigate this situation, but we expect no dependence for the other forms of the FENE-P model, because for $L^2 \geq 900$ the critical $Wi_k > 0.2$, and at $Wi_k = 0.2$, the extension of the dumbbells is not too high (for $\beta_p = 0.1$ and $L^2 = 900$ we have $C_{kk_{\text{max}}}/L^2 \approx 0.3$ and on increasing $L^2$, $C_{kk_{\text{max}}}/L^2$ decreases). However, as $L^2$ is lowered there could be an effect of the FENE-P model because at the critical $Wi_k$ the ratio $C_{kk_{\text{max}}}/L^2$ will be higher than at higher values of $L^2$, but this issue was not investigated in this work.

The computational cost of the semi-analytical solution at 1 s is negligible in comparison with that of the corresponding RheoFoam simulation at 4.5 h in a computer equipped with an Intel Xeon E5 processor with 12 MB L3 cache and Turbo Boost up to 3.9 GHz, with parallel processing using its six computer cores.

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