Short Communication

Effect of the solvent viscosity on pure electro-osmotic flow of viscoelastic fluids

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A B S T R A C T

In this work the fully-developed steady channel flow of the homogeneous polymer solution studied in [4] is revisited and a completely new analytical solution is proposed which is devoid of the limitations of the previous solution (\( \Phi D_e \leq 2/27 \)), i.e., it is valid for the complete range of the rheological parameters. The viscoelastic fluid is described by the simplified Phan–Thien and Tanner model with linear stress coefficient function for the polymer contribution plus a Newtonian solvent. The solution is also valid if the polymer contribution is described by the finitely extensive non-linear elastic model with the Peterlin approximation for the average spring force (FENE-P model).

1. Introduction

Forcing motion in microchannels through the application of an external electric field (electro osmotic flow - EOF) on fluids of complex rheology, many of which are engineered for specific applications, is becoming more common. Of particular interest are viscoelastic fluids, and even though their components may interact in a complex manner with the electric fields and surfaces, leading either to adsorption or wall-depletion, as described in the specialized literature [1,5,6], we are concerned here with the simple fully-developed steady channel flow of homogenous polymer solutions described by the simplified Phan–Thien and Tanner model with linear stress coefficient function for the polymer contribution plus a Newtonian solvent. This problem is of interest, for instance, to verify numerical methods in more complex situations and it was in this context that this flow was previously addressed in the appendix of Afonso et al. [4], where an analytical solution was originally proposed, but subject to constraints that limited its range of applicability, such as \( \Phi D_e \leq 2/27 \). In this short communication we present a completely new analytical solution devoid of such restrictions and then briefly discuss the effects of the solvent to polymer viscosity ratio and viscoelasticity upon the flow characteristics, since such an analysis is missing from the literature.

2. Analytical solution

We consider microfluidic transport of a viscoelastic fluid through a parallel plate microchannel (see Fig. 1) of height 2H, length L and width w, with w>>2H. The origin of the coordinate system is at the symmetry plane. An electric potential gradient is applied along the axis of the channel, which provides the necessary driving force for the flow. At the wall the no-slip condition applies and at the centreplane flow symmetry conditions apply (flow symmetry conditions are anti-symmetry of the shear stress, hence \( r_{xy} = 0 \) at \( y = 0 \)), so only one half of the channel \((0 < y < H)\) is considered for mathematical analysis (both walls are identical). To obtain a closed form solution for the velocity distribution, some simplifying assumptions are introduced: steady and fully developed flow, i.e., \( u = u(y) \), \( v = 0 \); no overlap of the spontaneously formed wall electric double layers, and validity of the Debye–Hückel linearization principle. Based on these assumptions the governing equations reduce to (for more details see the original work [4])

\[
V \cdot \tau = -\rho \varepsilon \mathbf{E} + \nabla p - \eta v^2 \mathbf{u},
\]

\[
f \left( \tau_{kk} \right) + \frac{v}{\lambda} \mathbf{D} = 2\eta \mathbf{D}.
\]

\[
V \cdot \left( \nabla \sigma \right) = -\frac{p_e}{\varepsilon}.
\]
Using Eq. (7), an explicit expression for the polymer normal stress component is obtained,
\[
\tau_{xx}^p = 2 \frac{A}{\eta_p} \left( \epsilon \zeta_0 \kappa \frac{\sinh(ky)}{\cosh(kH)} - \eta_p \frac{du}{dy} \right)^2.
\]

Note that Eq. (9) is consistent with the required boundary condition at the centerline, i.e. \( \tau_{xx}\big|_{y=0} = 0 \). From the simplification of Eq. (1) (see [4]), we obtain
\[
\frac{1}{1 + \frac{k^2\epsilon^2}{\eta_p^2} \left( \tau_{xx}^p \right)^2} \tau_{yy}^p = \frac{\eta_p}{\eta_s} \left( \tau_{yy}^0 + \epsilon \zeta_0 \kappa \frac{\sinh(ky)}{\cosh(kH)} \right) = 0.
\]

For the last equality on the right-hand side, the integrated momentum Eq. (8) was expressed explicitly in terms of the local shear rate \( du/dy \). After rearranging the various terms in Eq. (10) we arrive at the following cubic equation for the polymer shear stress
\[
\left( \tau_{xx}^p \right)^3 + \frac{\eta_p^2}{2k^2} \left( \tau_{yy}^0 + \epsilon \zeta_0 \kappa \frac{\sinh(ky)}{\cosh(kH)} \right) = 0,
\]
where \( \beta \) represents the viscosity ratio, \( n = \eta_s/\eta_p \). One may also think of \( 1 - \beta \) as a quantity proportional to the polymer concentration. The solution of this cubic equation is given in classical books as
\[
\tau_{xx}^p = \sqrt{-b + \sqrt{b^2 + 4a}} + \sqrt{-b - \sqrt{b^2 + 4a}}.
\]

In dimensionless form, the shear and normal stresses are given by:
\[
\tau_{xy}^p = \frac{2\beta \kappa d\varepsilon}{k(1 - \beta)} \tau_{xy}^0,
\]
respectively, where \( \tilde{y} = y/H, \tilde{k} = kH, \tilde{d} = \frac{2\kappa d}{\kappa d} = \kappa k \rho_{sh} \) is the Deborah number based on the EDL thickness and on the Helmholtz–Smoluchowski electro-osmotic velocity, defined as \( \rho_{sh} = \frac{\kappa k \rho_{sh}}{n} \) [9], and, \( \tau_{xy}^p = \frac{\tilde{\tau}_{xy}^p}{\kappa \rho_{sh}} \). The use of the Newtonian solvent viscosity coefficient to define the Helmholtz–Smoluchowski velocity and for normalization of the velocity profile is different from the previous works. The relationship between the Helmholtz–Smoluchowski velocities based on the solvent and polymeric part is \( u_{sh} = \kappa k \rho_{sh} \). It should be remarked that we are obtaining the polymeric stresses depending on the solvent through \( \beta \).

Going back to the momentum Eq. (8), we can re-write it as an equation for the dimensionless shear rate:
\[
\frac{d\tilde{\tau}_{xy}}{d\tilde{y}} = -\frac{k}{\kappa \rho_{sh}} \frac{\sinh(k\tilde{y})}{\cosh(k)}.
\]

In order to integrate Eq. (16), we assume the following variable transformation and approximation, \( \tilde{\tau}_{xy} = \sinh(k\tilde{y}) \approx \frac{1}{2} \exp(k \tilde{y}) - \frac{1}{2} \), but still it will be assessed in the next section. This approximation is usually accurate as in many practical applications the finite double layer thickness is very small, about 1–3 orders of magnitude smaller than the thickness of the microfluidic channel [3,4]. Note that in the previous works [3,4] the approximation used was \( \sinh(k\tilde{y}) \approx \frac{1}{2} \exp(k \tilde{y}) - \frac{1}{2} \) which was more accurate, avoiding at the same time the break of symmetry, reported in [3,4]. With \( \tilde{\tau}_{xy} \approx \frac{1}{2} \exp(k \tilde{y}) - \frac{1}{2} \), we have that:
\[
\frac{d\tilde{\tau}_{xy}}{d\tilde{y}} \approx \frac{1}{2} \exp(k \tilde{y}) + 1 \frac{d\tilde{\varphi}}{d\tilde{\varphi}}.
\]
where \( \tau_{y,\beta} \) is the shear stress of Eq. (14) with \( \delta = \frac{\omega(1-\beta)}{2 \cosh(\overline{\gamma})} \). If the integration is subject to the no-slip boundary condition at the wall (\( \overline{\partial}_{\text{wall}} = 0 \)), the resulting velocity profile is (now using again the inverse approximation \( \overline{\tau} = \sinh(\overline{x} \overline{y}) \approx \frac{1}{2} \exp(\overline{x} \overline{y}) - \frac{1}{2} \)),

\[
\overline{u}(\overline{y}) = \frac{1}{2} \operatorname{sech}(\overline{\gamma}) \left( \log \left( \frac{2 \sinh(\overline{x} \overline{y}) + 1}{2 \sinh(\overline{\gamma}) + 1} \right) - 2 \sinh(\overline{x} \overline{y}) + 2 \sinh(\overline{\gamma}) + (\overline{F} - \overline{F}^\prime)(1) - \overline{F}^\prime \right)
\]

where \( \overline{F} = \frac{\tau(1-\beta) \overline{F}^2}{(2 \cosh(\overline{y}))^2} \), and \( \overline{B} = \frac{1}{2 \overline{\gamma}} \left( \frac{1}{2} \overline{F}^2 \cosh^2(\overline{y}) \right)^{3/2} \) are constant parameters, and,

\[
\Omega(x) = \frac{1}{2} \sqrt{2/3} \sqrt{\overline{\gamma} \overline{\delta}} - \frac{a}{\overline{g}^{2/3}} \log \left( \sqrt{2/3} \sqrt{\overline{\gamma} \overline{\delta}} - \frac{\overline{\gamma} \overline{\delta} + \Omega^2}{\overline{\gamma} \overline{\delta} + \Omega^2} \right)
\]

\[
\overline{u}(\overline{y}) = \frac{1}{2} \operatorname{sech}(\overline{\gamma}) \left( \log \left( \frac{2 \sinh(\overline{x} \overline{y}) + 1}{2 \sinh(\overline{\gamma}) + 1} \right) - 2 \sinh(\overline{x} \overline{y}) + 2 \sinh(\overline{\gamma}) + (\overline{F} - \overline{F}^\prime)(1) - \overline{F}^\prime \right)
\]

This integral can be approximated using, for example, Simpson’s rule. If the domain is discretized into \( n \) (even number) equally spaced panels we obtain \( n + 1 \) grid points where the grid spacing is \( h = (1 - \overline{y})/n \). The approximation to the integral then becomes,

\[
\overline{u}(\overline{y}) \approx -\frac{h}{3} \left[ f(\overline{y}_0) + 2 \sum_{j=1}^{n-1} f(\overline{y}_j) + 4 \sum_{j=1}^{n} f(\overline{y}_{2j-1}) + f(\overline{y}_n) \right]
\]

where \( \overline{y}_0 = \overline{y}_n = 1 \) being the limits of integration, and \( f(\overline{y}) = -\frac{\tau(1-\beta)}{\cosh(\overline{y})} \). The error committed by the composite Simpson’s rule is bounded by

\[
\frac{h^4}{180}(b - a) \max_{x \in [a,b]} f''(x).
\]

The validity of the analytical velocity profile is tested as the influence of the solvent viscosity is assessed through the comparison between lines (analytical) and symbols (numerical). Note that the formulas obtained for the normalized shear and normal stresses are exact and do not depend on any approximation.

In the previous sections we derived the equations governing the steady fully developed EO flow of viscoelastic fluids in microchannels, taking into account the effect of the solvent viscosity. To better understand the corresponding flow dynamics, these equations are now presented via some plots by varying the different parameters involved.

We start by looking at Fig. 2 that shows the effects of the elasticity parameter \( cD_e \gamma^2 \), and of the ratio of viscosities \( \beta \), on the dimensionless velocity profiles for pure electro-osmotic flow. As shown before by Dhinakaran et al. [2] the width of the EDL decreases as \( \overline{\gamma} \) increases and the velocity profile becomes sharper near the wall. In the present solution the Debye–Hückel approximation is invoked, which requires \( \overline{\gamma} \gg 10 \), and in all plots we consider \( \overline{\gamma} = 20 \).

We observe the reduction of the normalized velocity profiles as the ratio of viscosities is decreased (increase of the polymeric concentration), consequence of the normalization of the Helmholtz–Smoluchowski viscosity, which is based exclusively on the solvent viscosity. This type of behavior was also reported by Cruz et al. [11], where two different normalizations had been applied to the velocity profiles obtained by analytic solution for fully developed pressure driven laminar pipe flow of PTT fluids with different solvent ratios. They reported two different outcomes: (1) a decrease of the velocity profile normalized by the average velocity as \( \beta \) decreases, similar to our results in Fig. 2(a), and (2) an increase of the velocity profile normalized by the Newtonian velocity based in pressure drop and total zero shear viscosity. As we increase \( cD_e \gamma^2 \) there is an increase of the dimensionless velocity, which is more intense for intermediate values of \( \beta \) at \( cD_e \gamma^2 = 16 \). This effect is better seen in Fig. 2(b), where we represent the maximum dimensionless velocity (velocity in the axis) vs. \( cD_e \gamma^2 \) for different viscosity ratios. Note that in this pure shear flow the coefficient \( cD_e \gamma^2 \) quantifies not only the elasticity via the normal stresses, but measures the shear-thinning of the viscosity that directly affects the velocity profile. An increase of the maximum velocity can be observed as we increase the ratio of the viscosities as well as the fluid shear-thinning. At large \( cD_e \gamma^2 \), the ratio \( (u/\Delta u)_\text{max} \) approaches a constant value (plateau), that is achieved more
rapidly for higher values of $\beta$. However for lower values of $\beta$ it is possible to see a delay of this effect, with the case of $\beta = 0.2$ showing an increasing in center plane velocity.

As expected, there is a very good agreement between the analytical and numerical solutions Fig. 2(a) and (b). We have considered different values of $\epsilon De_2^\ast$ and $\beta$.

The normalized flow rate can be visualized in Fig. 3(a), where the combination of both the effect of $\epsilon De_2^\ast$ and of the solvent ratio is presented. Care should be taken when analysing these results since due to the normalization adopted and the definition of $u_{sh}$, we obtain that $u_{sh} \to \infty$ as $\beta \to 0$. If this is not taken into account when analyzing Fig 3(a) and (b), one can wrongly infer that, for example, the flow rate goes to zero as we decrease $\beta$.

By looking at Fig. 3(a) we see that for lower $\epsilon De_2^\ast$ numbers, $\epsilon De_2^\ast = 0.25$, the flow rate decreases linearly with the reduction of $\beta$, and as we increase the $\epsilon De_2^\ast$ numbers the effect becomes non-linear, showing clearly an increase of the flow rate for the same solvent ratio. Another important aspect is the saturation effect of the polymeric concentration, i.e., as we reduce $\beta$ the different evolutions of the flow rate for the various $\epsilon De_2^\ast$ numbers converge to the same flow rate, reducing drastically the elastic impact, where at the limit for $\beta = 0.1$ the impact of the De numbers becomes negligible. The streaming electrical current, $I$, normalized by $\frac{e u_{sh}}{\kappa}$, also calculated through numerical integration of Eq. (22) by Simpson’s rule, is present in Fig. 3(b), showing similar evolution of the normalized flow rate, in terms of solvent viscosity and $\epsilon De_2^\ast$ variations, achieving the same saturation effect for $\beta = 0.1$.
Fig. 4 shows the profiles of shear and normal stresses as function of the near wall variable, \( x = (1 - \gamma) \kappa \), for different values of \( \epsilon D e^2_{\tau} \), drawn based on Eqs. (14) and (15). We can see in Fig. 4(a) that the shear stress approaches zero away from the wall (for \( x \geq 5 \)) for all values of \( \epsilon D e^2_{\tau} \) and increases rapidly near the wall, being more intense for lower values \( \epsilon D e^2_{\tau} \) and higher values of \( \beta \). The saturation effect is also observed at lower values of \( \beta \) were the elastic contribution has a low impact, this can be observed for \( \beta = 0.2 \), where the shear stress profiles practically coincide for the all range of \( \epsilon D e^2_{\tau} \) presented.

The results obtained for the normal stress are qualitatively similar to the ones obtained for the shear stress, with the difference that the normal stress is positive (Fig. 4(b)). Note that a relationship between shear and normal stresses was established by Eq. (14), and therefore, as we approach to the wall the normal stress increases with more intensity for lower \( \epsilon D e^2_{\tau} \) numbers and higher values of solvent ratio.

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