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Analytical solutions for channel flows of Phan-Thien–Tanner and Giesekus fluids under slip

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ABSTRACT

Analytical and semi-analytical solutions are presented for the cases of channel and pipe flows with wall slip for viscoelastic fluids described by the simplified PTT (using both the exponential and the linearized kernel) and the Giesekus models. The slip laws used are the linear and nonlinear Navier, the Hatzikiriakos and the asymptotic models. For the nonlinear Navier slip only natural numbers can be used for the exponent of the tangent stress in order to obtain analytical solutions. For other values of the exponent and other nonlinear laws a numerical scheme is required, and thus, the solution is semi-analytical. For these cases the intervals containing the solution and the corresponding proof for the existence and uniqueness are also presented. For the Giesekus model the influence of the wall slip on the restrictions of the slip models is also investigated.

1. Introduction

Analytical solutions are a valuable tool to understand the complexity of fluid dynamics. The Cauchy equation together with a rheological constitutive equation, allow the determination of the flow characteristics of non-Newtonian fluids. However, these are complex equations for which analytical solutions can only be obtained for basic flows in simple geometries. Adding slip boundary conditions to this system of equations increases the complexity to obtain analytical solutions.

Understanding the influence of slip on the flow behavior is crucial to comprehend some characteristics of industrial flows relevant for the polymer processing industry [1]. The mathematical study of Navier slip boundary conditions for Stokes fluids was carried out by Fujita [2], who was only concerned with the wellposedness of the system of equations. Mitsoulis and Hatzikiriakos [3] have studied the application of these slip boundary conditions to polymer extrusion using generalized Newtonian fluids. Later they presented some analytical solutions for lubrication flows in convergent channels and compared them with the corresponding numerical results [4]. In this way they could identify the conditions for validity of the analytical solution obtained using the lubrication theory, for different degrees of contraction.

For viscoelastic materials described by a differential stress constitutive equation, published work using slip boundary conditions is scarce. Here, Pereira [5] studied microfluidic flows under slip of Newtonian, generalized Newtonian and viscoelastic fluids governed by the linearized White–Metzner model using the Navier slip boundary condition.

For the simplified Phan-Thien—Tanner (PTT) and Giesekus models no analytical solutions with slip boundary conditions have been reported in the literature, but there are several analytical solutions in the absence of wall slip. For the PTT fluid we single out the solutions for Couette flow [6–8], and for channel and pipe flows [9]. For the Giesekus model solutions exist for no slip channel and pipe flows with the inclusion of a solvent contribution [10] as well as without solvent [11]. There are also analytical solutions for no slip planar Couette–Poiseuille flow [12], concentric annular flow [13,14] and Taylor–Couette flow with inner cylinder rotation [15].

The aim of this work is then, to fill the gap of analytical solutions for Couette and Poiseuille flows of viscoelastic fluids described by the simplified PTT and Giesekus constitutive equations considering slip velocity at the wall.

The paper is organized as follows: first, in Section 2, the governing equations are presented for both constitutive models and the various slip models used are also presented and simplified for the simple case of flow between parallel plates. These slip laws are the linear and the nonlinear Navier, the Hatzikiriakos and the asymptotic slip models. In Section 3 analytical solutions are given for the Couette and the Poiseuille flows of a PTT fluid under various conditions for the selected slip laws. For some cases like the Navier

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slip law, it is possible to present an analytical solution for the inverse problem (where the pressure gradient is computed as a function of the average velocity), but for the remaining cases the numerical solution of an equation is required (semi-analytical solution). In Section 4, the Giesekus model [16] is considered and again the Couette and Poiseuille flows are studied for the various slip laws. Section 5 concludes/summarizes the main findings of this work.

2. Governing equations

It is assumed that the fluid is incompressible and governed by the continuity (Eq. (1)) and momentum (Eq. (2)) equations,

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \rho \nabla \cdot \mathbf{u}\mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{\tau}, \tag{2}$$

together with a constitutive equation for the stress $\boldsymbol{\tau}$. In Eqs. (1) and (2), $\mathbf{u}$ is the velocity vector, p is the pressure and $\boldsymbol{\tau}$ is the deviatoric stress tensor.

The simplified PTT constitutive model is given by the following equation,

$$f(tr\boldsymbol{\tau})\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} = \eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}), \tag{3}$$

where $f(tr\boldsymbol{\tau})$ is a function depending on the trace of the stress tensor, λ is the relaxation time, η is the viscosity coefficient and $\overset{\triangledown}{\boldsymbol{\tau}}$ stands for Oldroyd's upper convective derivative (Eq. (4)),

$$\overset{\triangledown}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} - [(\nabla \mathbf{u})^{\mathrm{T}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \nabla \mathbf{u}]. \tag{4}$$

The function $f(tr\boldsymbol{\tau})$ can take the form of the exponential equation [17],

$$f(tr\boldsymbol{\tau}) = \exp\left(\frac{\varepsilon\lambda}{\eta} tr\boldsymbol{\tau}\right), \tag{5}$$

as well as the linearized function (Eq. (6)), presented by [18],

$$f(tr\boldsymbol{\tau}) = 1 + \frac{\varepsilon\lambda}{\eta} tr\boldsymbol{\tau}. \tag{6}$$

Parameter ε is inversely proportional to the extensional viscosity of the fluid and the linearized function only approaches well the exponential form at low deformations.

The Giesekus constitutive model is given by,

$$\boldsymbol{\tau} + \frac{\alpha\lambda}{\eta}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}) + \lambda \overset{\triangledown}{\boldsymbol{\tau}} = \eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}), \tag{7}$$

where α is the so-called mobility parameter. This model is based on molecular concepts and it reproduces well many of the characteristics of polymeric fluids [16].

Considering a Cartesian coordinate system with x, y, z in the streamwise, transverse and spanwise directions, respectively, and since the flows studied in this work are the fully developed Couette and Poiseuille flows (cf. Fig. 1), the governing equations can be simplified because,

$$\partial/\partial x = 0 \text{ (except for pressure)}, \quad \partial v/\partial y = 0, \quad \partial p/\partial y = 0. \tag{8}$$

This implies the automatic satisfaction of the continuity equation, whereas the momentum equation simplifies and can be integrated to become,

$$\tau_{xy} = p_x y + c_1, \tag{9}$$

where p_x stands for the pressure gradient in the x direction, τ_{xy} is the shear stress and c_1 is a stress constant. Eq. (9) is valid regardless of the rheological constitutive equation.

The simplified forms of the constitutive equations for the fully developed flow conditions are somewhat different and they will be presented later, at the beginning of the corresponding results section.

The slip boundary conditions investigated here are the linear and nonlinear Navier, the Hatzikiriakos and the asymptotic slip laws.

For the **nonlinear Navier slip law** [19] the nonlinear power function relating wall shear stress and wall slip is given by following equation,

$$u_w = (\mp \tau_{xy,w})^m k, \tag{10}$$

where $m > 0 (m \in \mathbb{R})$. When $m = 1$ the Navier linear slip law [20] is recovered. The signs $\mp$ stand for the upper $-$ and bottom + walls, assuming there is flow between the parallel plates and the coordinate system is given as in Fig. 1.

Hatzikiriakos [21] proposed a slip model based on Eyring's theory of liquid viscosity that provides a smooth transition from no-slip to slip flow at the critical shear stress τ_c (positive constant). The one dimensional **Hatzikiriakos slip law** is given by,

$$u_w = \begin{cases} k_{H1} \sinh(\mp k_{H2} \tau_{xy,w} - \tau_c), & \text{if } |\tau_{xy}| \geqslant \tau_c, \\ 0, & \text{if } |\tau_{xy}| < \tau_c, \end{cases} \tag{11}$$

where $k_{H1}, k_{H2} \in [0, +\infty[$ are the friction coefficients. In this work we have considered only a null critical stress ($\tau_c = 0$).

The last slip model investigated here is the asymptotic slip law [22], given for one dimensional flow by,

$$u_w = k_{A1} \ln(\mp k_{A2} \tau_{xy,w} + 1), \tag{12}$$

with $k_{A1}, k_{A2} \in [0, +\infty[$.

3. Analytical solutions for the PTT fluid and discussion

For the fully developed Couette and Poiseuille flows (cf. Fig. 1), the system of rheological constitutive equations for the simplified PTT model is given by,

$$f(\tau_{xx} + \tau_{yy})\tau_{xx} = 2\lambda \tau_{xy}(\partial u/\partial y), \tag{13a}$$

$$f(\tau_{xx} + \tau_{yy})\tau_{yy} = 0, \tag{13b}$$

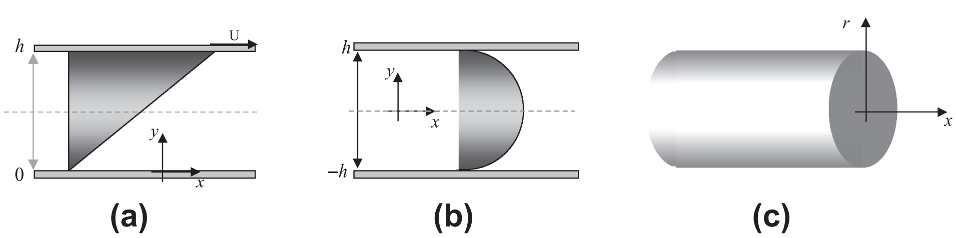

Fig. 1. Geometry of the Couette (a), Poiseuille planar channel (b) and Poiseuille pipe (c) flows.

$$f(\tau_{xx}+\tau_{yy})\tau_{xy}=\eta(\partial u/\partial y)+\lambda\tau_{yy}(\partial u/\partial y), \tag{13c}$$

From Eq. (13b) one can see that $f(\tau_{xx}+\tau_{yy})=0$ or $\tau_{yy}=0$, but if $f(\tau_{xx}+\tau_{yy})=0$, unrealistic results would be obtained hence the only possible solution is $\tau_{yy}=0$.

Dividing Eq. (13a) by Eq. (13c), the former becomes $\tau_{xx}=2\lambda/\eta(\tau_{xy})^2$. If Eqs. (13a–c) are combined with the momentum equation, the following system is obtained,

$$\tau_{xy}=p_x y+c_1, \tag{14a}$$
$$\tau_{xx}=(2\lambda/\eta)(\tau_{xy})^2, \tag{14b}$$
$$\tau_{yy}=0, \tag{14c}$$
$$f(\tau_{xx}+\tau_{yy})\tau_{xy}=\eta(\partial u/\partial y), \tag{14d}$$

Length, velocity and stresses are scaled with h, U and $\eta U/h$, respectively (U is the mean streamwise velocity), leading to the dimensionless system of equations in Eq. (15), with $y'=y/h, u'(y')=u(y')/U, c_1'=c_1/(\eta U/h)$ and $\tau_{xx}'=\tau_{xy}/(\eta U/h)$,

$$\tau_{xy}'=p_x'y'+c_1', \tag{15a}$$
$$\tau_{xx}'=2Wi(p_x'y'+c_1')^2, \tag{15b}$$
$$\tau_{yy}'=0, \tag{15c}$$
$$(\partial u'/\partial y')=f[\tau_{xx}'](p_x'y'+c_1'), \tag{15d}$$

together with $f[\tau_{xx}']=1+2\varepsilon Wi^2(p_x'y'+c_1')^2$ for the linear PTT and the function $f[\tau_{xx}']=\exp(2\varepsilon Wi^2(p_x'y'+c_1')^2)$ for the exponential PTT. In the previous expressions $Wi=\lambda U/h$ is the Weissenberg number.

The boundary conditions are written in a dimensionless form for Couette flow in Eqs. (16a–c) for the nonlinear Navier, the Hatzikiriakos and the asymptotic slip laws, respectively,

$$u_w'(0)=k_m'(c_1')^m, \tag{16a}$$
$$u_w'(0)=k_{H1}'\sinh(k_{H2}'c_1'), \tag{16b}$$
$$u_w'(0)=k_{A1}'\ln(1+k_{A2}'c_1'), \tag{16c}$$

and correspondingly by Eqs. (17a–c) for Poiseuille flow

$$u_w'(\pm 1)=k_m'(-p_x')^m, \tag{17a}$$
$$u_w'(\pm 1)=k_{H1}'\sinh(k_{H2}'p_x'), \tag{17b}$$
$$u_w'(\pm 1)=k_{A1}'\ln(1-k_{A2}'p_x'), \tag{17c}$$

where $k_m'=kU^{m-1}(\eta/h)^m, k_{H1}'=k_{H1}/U, k_{H2}'=k_{H2}\eta U/h, k_{A1}'=k_{A1}/U, k_{A2}'=k_{A2}\eta U/h\in\mathbb{R}_0^+, m\in\mathbb{R}^+$.

3.1. Couette flow – linear and exponential PTT models

For the Couette flow (Fig. 1a) with slip velocity at the moving wall, the only admissible solution for the velocity profile is the trivial solution $u'(y')=0$ [23], regardless of the boundary condition at the immobile wall.

For the Couette flow with slip velocity at the immobile wall and no slip at the moving wall and since the pressure gradient is null (by Eq. (15a) the shear stress is constant c_1') the system of equations simplifies to following equation,

$$\tau_{xy}'=c_1', \tag{18a}$$
$$\tau_{xx}'=2Wi(c_1')^2, \tag{18b}$$
$$\tau_{yy}'=0, \tag{18c}$$
$$(\partial u'/\partial y')=f[c_1']c_1' \tag{18d}$$

with $f[c_1']=1+2\varepsilon Wi^2(c_1')^2$ for the linear PTT and $f[c_1']=\exp(2\varepsilon Wi^2(c_1')^2)$ for the exponential PTT.

Integrating Eq. (18d) and applying the Dirichlet boundary condition at the upper wall,

$$u_w'(1)=1, \tag{19}$$

together with one of the slip boundary conditions (Eqs. (16a–c)) at the lower wall, the velocity profile $u'(y')$ and c_1' are given by Eqs. (20) and (21) for the linear PTT model

$$u'(y')=[c_1'+2\varepsilon Wi^2(c_1')^3](y')+u_w'(0), \tag{20}$$
$$C_1'+2\varepsilon Wi^2(c_1')^3+u_w'(0)-1=0, \tag{21}$$

and by Eqs. (22) and (23) for the exponential PTT model

$$u'(y')=\exp(2\varepsilon Wi^2(c_1')^2)c_1'y'+u_w'(0), \tag{22}$$
$$\exp(2\varepsilon Wi^2(c_1')^2)c_1'+u_w'(0)-1=0. \tag{23}$$

Due to nonlinearities, the full analytical solutions are obtained only for the following few cases: the linear PTT model with Navier slip law and exponents $m=1, 2, 3$, and the exponential PTT with no slip velocity.

For the linear PTT with $m=1$, we have that,

$$c_1'=\left((4\varepsilon Wi^2)^{-1}+\sqrt{((4\varepsilon Wi^2)^{-1})^2+\left(\frac{1+k_m'}{6\varepsilon Wi^2}\right)^3}\right)^{1/3}+\left((4\varepsilon Wi^2)^{-1}-\sqrt{((4\varepsilon Wi^2)^{-1})^2+\left(\frac{1+k_m'}{6\varepsilon Wi^2}\right)^3}\right)^{1/3} \tag{24}$$

whereas for exponential PTT with $k_m'=0, c_1'$ is given by,

$$c_1'=[2(\varepsilon Wi^2/\mathbf{W}(4\varepsilon Wi^2))^{0.5}]^{-1}. \tag{25}$$

Substitution of Eqs. (24) and (25) on the expressions for the velocity profile Eqs. (20) and (22) (for the linear and exponential PTT, respectively) gives the final solution. Note that the latter solution depends on the Lambert function $\mathbf{W}$, that can be expressed as the solution of Eq. (26).

$$\mathbf{W}(x)e^{\mathbf{W}(x)}=x. \tag{26}$$

These results show that the stress c_1' will be influenced by the presence of slip. The analytical solutions for the nonlinear Navier slip model with $m=2, 3$ can be found in Appendix A, which includes the proof for the existence and uniqueness of solutions for other values of m and for the Hatzikiriakos and asymptotic slip models, together with the corresponding interval where the solutions are located.

The relationship between slip velocity, stress and εWi^2 was studied for both PTT models with linear Navier slip law, and is plotted in Fig. 2.

As the slip velocity increases to total slip, the dimensionless shear stress decreases to zero, regardless of the slip model and Weissenberg number (for full slip conditions, the velocity profile is a plug flow since there is no shear and the normal stresses are null).

Lower shear stresses are obtained for the exponential PTT when compared with the linear PTT especially as the no-slip condition is approached. As slip increases the shear rates are smaller and under these conditions the linear stress function (first two terms of a Taylor expansion) approaches well the exponential stress function. It is also shown that the shear stress decreases with the increase of Wi on account of shear thinning behavior.

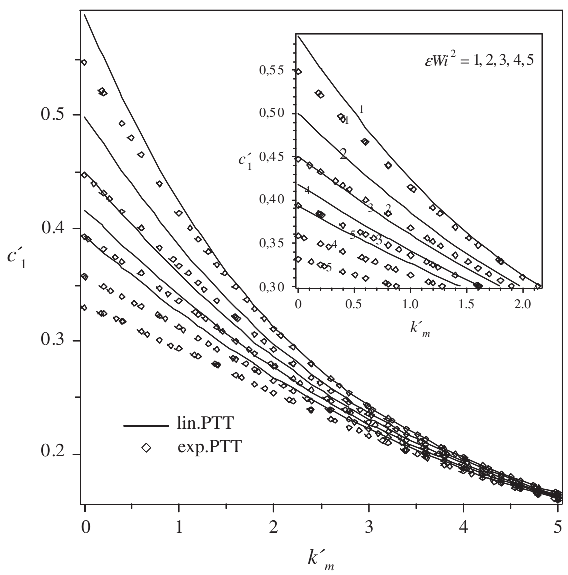

Fig. 2. Variation of c'_1 as a function of slip k'_m and εWi^2 for Couette flow with slip at the fixed wall and no-slip at the moving wall. The εWi^2 numbers are given next to the graph in the zoomed view.

3.2. Planar channel flow with the linear PTT model

For the Poiseuille flow (Fig. 1b) it is assumed that the same boundary condition is applied at the top and bottom walls leading to a symmetric flow, hence from Eq. (15a) $c'_1 = 0$.

From Eq. (15d) one obtains,

$$(\partial u'/\partial y') = p'_x y' + 2\varepsilon Wi^2 (p'_x y')^3, \tag{27}$$

that after integration gives,

$$u'(y') = 0.5 p'_x y'^2 + 0.5\varepsilon Wi^2 (p'_x)^3 y'^4 + c, \ \ c \in \mathbb{R}, \tag{28}$$

where there are two unknowns, the pressure gradient p'_x and c. In order to obtain a unique solution and determine c, a boundary condition given by any of the Eqs. (17a-c) must be provided, here represented by $u'_w(1)$. The velocity profile is then given by the following equation,

$$u'(y') = 0.5 p'_x (y'^2 - 1) + 0.5\varepsilon Wi^2 (p'_x)^3 (y'^4 - 1) + u'_w(1). \tag{29}$$

By applying a constant flow rate $Q = Uh$ (with U the imposed average velocity) and integrating Eq. (29) over half of the channel width, the following equation is achieved for the pressure gradient,

$$\int_0^1 u'(y')dy' = 1 \Rightarrow (-2/5)\varepsilon Wi^2 (p'_x)^3 + p'_x(-1/3) - 1 + u'_w(1) = 0 \tag{30}$$

The nonlinearity of Eq. (30) ($u'_w(1)$ depends on p'_x, cf. Eqs. (17a–c) reduces the existence of full analytical solutions to just a few cases, $m = 1, 2, 3$.

Assuming $m = 1$ in Eq. (17a), Eq. (30) can be rewritten after some algebra as,

$$(p'_x)^3 + p'_x \underbrace{\frac{(1/3 + k')}{(2/5)\varepsilon Wi^2}}_{R} + \underbrace{\left((2/5)\varepsilon Wi^2\right)^{-1}}_{Q} = 0 \tag{31}$$

According to the Cardano–Tartaglia formula [24] this cubic equation has the following real solution for the pressure gradient as a function of the imposed flow rate,

$$p'_x = \left(-Q/2 + [(Q/2)^2 + (R/3)^3]^{1/2}\right)^{1/3} + \left(-Q/2 - [(Q/2)^2 + (R/3)^3]^{1/2}\right)^{1/3}, \tag{32}$$

with R and Q defined in Eq. (31).

With this explicit formula, the velocity profile (Eq. (29)) will no longer depend on the pressure gradient, and it can be written (Eq. (33)) as a function of the y' coordinate (assuming all the parameters are known),

$$u'(y') = \left((a+b)^{1/3} + (a-b)^{1/3}\right)[0.5(y'^2 - 1) - k'] + \left((a+b)^{1/3} + (a-b)^{1/3}\right)^3 0.5\varepsilon Wi^2 (y'^4 - 1), \tag{33}$$

where

$$(a \pm b)^{1/3} = \left(-((4/5)\varepsilon Wi^2)^{-1} \pm \left[((4/5)\varepsilon Wi^2)^{-2} + \left(\frac{1}{3}\frac{(1/3 + k')}{(2/5)\varepsilon Wi^2}\right)^3\right]^{1/2}\right)^{1/3}.$$

For $m = 2$, Eq. (30) can be rearranged and rewritten as Eq. (34),

$$(p'_x)^3 + \frac{k'_m}{(-2/5)\varepsilon Wi^2}(p'_x)^2 + \frac{(1/3)}{(2/5)\varepsilon Wi^2} p'_x + [(2/5)\varepsilon Wi^2]^{-1} = 0 \tag{34}$$

The solution of the Cardano–Tartaglia formula shows that the real roots of this cubic equation are different.

For $m = 3$, the cubic Eq. (35) for the pressure gradient is similar to Eq. (31) and its real solution is also given by Eq. (32), with the new definitions of R and Q.

$$(p'_x)^3 + p'_x \underbrace{\frac{(1/3)}{(2/5 + k'_m)\varepsilon Wi^2}}_{R} + \underbrace{[(2/5 + k'_m)\varepsilon Wi^2]^{-1}}_{q} = 0 \tag{35}$$

Hence, the velocity profile can be computed by following equation,

$$u'(y') = \left((a+b)^{1/3} + (a-b)^{1/3}\right)[0.5(y'^2 - 1)] + \left((a+b)^{1/3} + (a-b)^{1/3}\right)^3 0.5\varepsilon Wi^2 (y'^4 - 1 - k'_m) \tag{36}$$

where $(a \pm b)^{1/3} = \left(-\left((4/5 + k'_m)\varepsilon Wi^2\right)^{-1} \pm \left[\left((4/5 + k'_m)\varepsilon Wi^2\right)^{-2} + \left(\frac{1}{3}\frac{(1/3)}{(2/5 + k'_m)\varepsilon Wi^2}\right)^3\right]^{1/2}\right)^{1/3}.$

For the Hatzikiriakos and asymptotic models and for other nonlinear slip exponents, the solution is semi-analytical and requires a procedure like the one adopted in Appendix A. Incidentally, for $m = 4$ it is still possible to obtain a closed form analytical solution. In the Supplementary material appended to this work we give the solution for the pressure gradient equation (Eq. (34)) for the four different slip boundary conditions and for different values of $\varepsilon Wi^2, k'_m, k'_{H1}, k'_{H2}, k'_{A1}$ and k'_{A2}.

3.3. Planar channel flow with the exponential PTT model

Eq. (15d) for the exponential PTT model (Eq. (5)) and considering symmetry on the centreplane leads to

$$\frac{\partial u'}{\partial y'} = \exp(2\varepsilon Wi^2 (p'_x y')^2) p'_x y'. \tag{37}$$

After integration and application of the boundary condition $u'_w(1)$ the velocity profile is

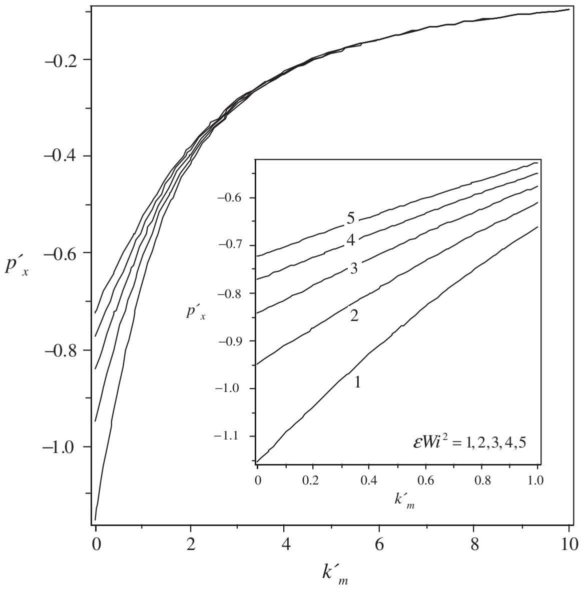

Fig. 3. Variation of p'_x as a function of k'_m and εWi^2 for a Poiseuille planar channel flow with the linear PTT model and linear Navier slip. The εWi^2 numbers are given next to each graph in the zoomed view.

$$u'(y') = [4\varepsilon Wi^2(p'_x)]^{-1}(\exp(2\varepsilon Wi^2(p'_x y')^2) - \exp(2\varepsilon Wi^2(p'_x)^2)) + u'_w(1) \tag{38}$$

where $u'_w(1)$ is the boundary condition given by any of the Eqs. (17a–c).

The solution of the inverse problem is achieved as for the linear PTT model, i.e. integrating the velocity profile of Eq. (38) now leading to,

$$\int_0^1 \exp(2\varepsilon Wi^2(p'_x y')^2)dy' = \exp(2\varepsilon Wi^2(p'_x)^2) + 4\varepsilon Wi^2(p'_x) - 4\varepsilon Wi^2(p'_x)u'_w(1). \tag{39}$$

and then solving in order to the pressure gradient. To evaluate the left hand side (lhs) of Eq. (39) use is made of the definition of the error function (*erf*), giving

$$-\frac{i\sqrt{\pi}}{2\sqrt{G}} erf\left(i\sqrt{2\varepsilon Wi^2(p'_x)^2}\right) = \exp(2\varepsilon Wi^2(p'_x)^2) + 4\varepsilon Wi^2(p'_x) - 4\varepsilon Wi^2(p'_x)u'_w(1). \tag{40}$$

Eq. (40) can be further simplified and written as,

$$\sum_{k=0}^{\infty} \frac{(2\varepsilon Wi^2(p'_x)^2)^k}{(2k+1)k!} = exp(2\varepsilon Wi^2(p'_x)^2) + 4\varepsilon Wi^2(p'_x) - 4\varepsilon Wi^2(p'_x)u'_w(1). \tag{41}$$

This series is convergent and since it calculates the area under a known function, it can be shown that the lhs of Eq. (41) is a monotonic function in the range $p'_x \in]-\infty, 0]$. To obtain the pressure gradient the range containing the solution must be known and the bisection method is then applied. Care must be taken because of the sharp changes occurring while changing the slip friction coefficient.

Tables are given as Supplementary material containing the solution for the pressure gradient equation (Eq. (41)) for the four different slip boundary conditions and for different values of $\varepsilon Wi^2, k'_m, k'_{H1}, k'_{H2}, k'_{A1}$ and k'_{A2}.

The results for the Poiseuille flow with the linear and exponential PTT models can be summarized as follows. For the linear PTT model with linear Navier slip, the absolute value of the pressure drop decreases (tends to zero) with the increase of both slip and εWi^2, as observed in Fig. 3. So, the effect of slip on p'_x is as in Couette flow. An increase in εWi^2 increases the shear rate, while imparting shear-thinning behavior to the fluid so that ultimately it reduces significantly its shear viscosity as shown in Fig 4a in terms of the shear stress τ'_{xy}. The corresponding normal stress τ'_{xx} variation is shown in Fig. 4b and is similar to that of the shear stress since they are proportional. In the absence of slip the results match those of Oliveira and Pinho [9].

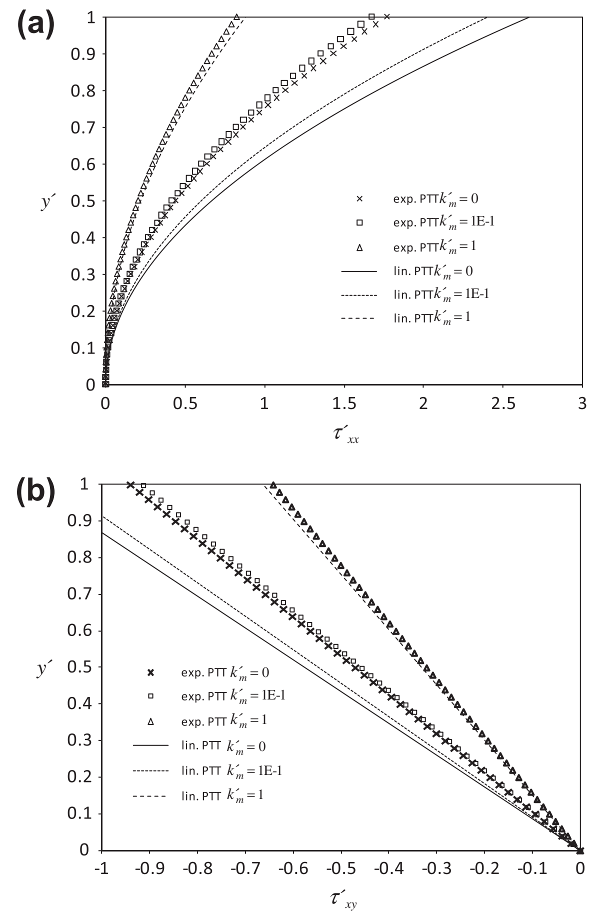

Fig. 4. Variation of τ'_{xx} (a) and τ'_{xy} (b) along the channel half width y' for a Poiseuille flow of a PTT model with different values of k'_m and constant $\varepsilon Wi^2 = 1$.

As in the Couette flow, and for the same reasons, the exponential PTT model exhibits lower stresses than the corresponding linear PTT model.

The solution for the Poiseuille pipe flow of a PTT fluid (linear and exponential) is given in Appendix B.

4. Analytical solutions for the Giesekus fluid and discussion

The derivation of the equations is well explained by Yoo and Choi [10], Schleiniger and Weinacht [11] and Raisi et al. [12] and here we follow the same sequence as in Section 3 for the PTT model.

Based on the simplifications for fully developed flow in the geometries of Fig. 1, the dimensionless momentum and constitutive equations become (h, U and $\eta U/h$ are the length, velocity and stress scales, respectively),

$$\frac{\partial \tau'_{xy}}{\partial y'} = p'_x \tag{42a}$$

$$2De\tau'_{xy}\frac{\partial u'}{\partial y'} = \tau'_{xx} + \alpha Wi(\tau'^2_{xx} + \tau'^2_{yy}) \tag{42b}$$

$$(1 + Wi\tau'_{yy})\left(\frac{\partial u'}{\partial y'}\right) = \tau'_{xy} + \alpha Wi\tau'_{xy}(\tau'_{xy} + \tau'_{xx}) \tag{42c}$$

$$\alpha Wi(\tau'^2_{xy} + \tau'^2_{yy}) + \tau'_{yy} = 0 \tag{42d}$$

where $Wi = \lambda U/h$ is the Weissenberg number.

Redefining dimensionless quantities as $u^* = Wiu'$, $\tau^*_{xx} = Wi\tau'_{xx}$, $\tau^*_{yy} = Wi\tau'_{yy}$ and $\tau^*_{xy} = Wi\tau'_{xy}$ the previous system of equations (Eq. (42)) can be integrated and presented as in the following equation,

$$\tau^*_{xy} = Wi(p'_x y' + c'_2) \tag{43a}$$

$$\frac{\partial u^*}{\partial y'} = 2\alpha\tau^*_{xy}\frac{\left(1 \pm (2\alpha-1)\sqrt{1-4\alpha^2(\tau^*_{xy})^2}\right)}{(2\alpha - 1 \pm \sqrt{1-4\alpha^2(\tau^*_{xy})^2})^2} \tag{43b}$$

$$\tau^*_{xx} = \frac{\left[(1-\alpha)(1 \mp \sqrt{1-4\alpha^2(\tau^*_{xy})^2}) + 2\alpha^2(\tau^*_{xy})^2\right]}{\alpha(2\alpha - 1 \pm \sqrt{1-4\alpha^2(\tau^*_{xy})^2})} \tag{43c}$$

$$\tau^*_{yy} = \frac{(-1 \pm \sqrt{1-4\alpha^2(\tau^*_{xy})^2})}{2\alpha} \tag{43d}$$

Eqs. (43b,c) require $1 - 4\alpha^2(\tau^*_{xy})^2 > 0$ and in addition, thermodynamic considerations [10] require a positive first normal stress difference ($\tau^*_{xx} - \tau^*_{yy} \geqslant 0$). This inequality can be further simplified [10] leading to $2\alpha - 1 \pm \sqrt{(1-4\alpha^2(\tau^*_{xy})^2)} > 0$.

These restrictions on the system of equations bring two sets of solutions [10], the so-called upper branch solution, where

$$\begin{cases} \tau^*_{xy} < \sqrt{1/\alpha - 1} & \text{for } \alpha \in]0;1/2] \\ \tau^*_{xy} \leqslant 1/2\alpha & \text{for } \alpha \in]1/2;1] \end{cases} \tag{44}$$

and the lower branch solution for which

$$\sqrt{1/\alpha - 1} < \tau^*_{xy} \leqslant 1/2\alpha \quad \text{for } \alpha \in]1/2;1]. \tag{45}$$

These restrictions imply that for some values of the Weissenberg number the solutions for Couette and Poiseuille flows with no slip may not exist as already shown by Yoo and Choi [10]. Next we analyse the cases with slip but since the lower branch solution presents physically unrealistic solutions [10] only the upper branch solution needs to be considered.

4.1. Couette flow

Following Yoo and Choi [10] let $\psi = 2\alpha\tau^*_{xy}$, $p'_x = 0$ in Couette flow, so ψ can be written as $\psi = 2\alpha Wic'_2$ (with the help of Eq. (43a)) and integrate Eq. (43b) to obtain the following velocity profile

$$u^*(y') = \frac{\psi\left(1 \pm (2\alpha-1)\sqrt{1-\psi^2}\right)}{\left(2\alpha - 1 \pm \sqrt{1-\psi^2}\right)^2}y' + c. \tag{46}$$

where c is the constant of integration determined by the lower wall slip boundary conditions with the proper normalization (Eqs. (16a–c)) multiplied by Wi (Eqs. (47a–c)). These are given by,

$$u^*_w(0) = Wik'_m\left(\frac{\psi}{2\alpha Wi}\right)^m, \tag{47a}$$

for the (non)linear Navier slip law,

$$u^*_w(0) = Wik'_{H1}\sinh\left(k'_{H2}\frac{\psi}{2\alpha Wi}\right), \tag{47b}$$

for the Hatzikiriakos slip law, and

$$u^*_w(0) = Wik'_{A1}\ln\left(1 - k'_{A2}\frac{\psi}{2\alpha Wi}\right), \tag{47c}$$

for the asymptotic slip law.

At the upper wall there is no slip (cf. Section 3.1 for the justification) and the Dirichlet boundary condition is

$$u^*_w(1) = Wi, \tag{48}$$

and this condition together with Eq. (46) provides the following implicit equation relating the Weissenberg number and c'_2 (note that $\Psi = 2\alpha Wic'_2$)

$$2\alpha Wic'_2\frac{\left(1 \pm (2\alpha-1)\sqrt{1-(2\alpha Wic'_2)^2}\right)}{\left(2\alpha - 1 \pm \sqrt{1-(2\alpha Wic'_2)^2}\right)^2} + u^*_w(0) - Wi = 0. \tag{49}$$

This equation must be solved numerically with the following restriction on Wi,

$$Wi \cdot c'_2 \leqslant \frac{1}{2\alpha}. \tag{50}$$

For the special case of the linear Navier slip law it is possible to analytically find the limiting admissible values for Wi and c'_2. Based on the definition of ψ, the upper branch solution (Eq. (44)) can be rewritten as,

$$\begin{cases} \psi < 2\alpha\sqrt{1/\alpha - 1} & \text{for } \alpha \in]0;1/2], \\ \psi \leqslant 1 & \text{for } \alpha \in]1/2;1], \end{cases} \tag{51}$$

Eq. (49) with the Navier slip boundary condition can be rewritten as,

$$Wi(\psi) = \frac{\psi\left(1 \pm (2\alpha-1)\sqrt{1-\psi^2}\right)}{\left(2\alpha - 1 \pm \sqrt{1-\psi^2}\right)^2} + \frac{\psi k'_m}{2\alpha}, \tag{52}$$

$$Wi(\psi) = \frac{\partial u^*}{\partial y'} + \frac{\psi k'_m}{2\alpha}. \tag{53}$$

Physical reasons require the solution to verify $\partial(\partial u^*/\partial y)/\partial\tau^*_{xy} = 2\alpha\partial(\partial u^*/\partial y)/\partial\psi > 0$ [10], i.e., shear rate increases with shear stress and this is verified by the upper branch solution. Since $\partial(\partial u'/\partial y)/\partial\Psi > 0$ and $\partial(\Psi k'_m/2\alpha)/\partial\Psi > 0$, by Eq. (53) $\partial(Wi(\psi))/\partial\psi > 0$ meaning that $Wi(\psi)$ is a monotonically increasing function of ψ. Thus, for $\alpha \in]1/2;1]$ and considering Eq. (52), the restrictions are given by,

$$\lim_{\psi\to 1} Wi(\psi) = \frac{1}{(2\alpha-1)^2} + \frac{k'_m}{2\alpha}, \tag{54}$$

for the Weissenberg number and by

$$\left(\frac{2\alpha}{(2\alpha-1)^2} + k'_m\right)^{-1} \leqslant c'_2 < 1. \tag{55}$$

for the stress coefficient c'_2 (obtained combining Eqs. (52)–(54). Note that $c'_2 < 1$ because the fluid is shear thinning [10]. For the range $\alpha \in]0; 1/2]$ there are no restrictions.

As shown by Eqs. (54) and (55), increasing the slip velocity smoothes the restriction on Wi and c'_2. For the other slip laws, it is more difficult to determine these restrictions on the Weissenberg number and stress constants. Although those slip laws

Table 1
Minimum and maximum admissible values for c_2' and Wi as a function of α and k_m' for Couette flow of Giesekus model with the linear Navier slip law.

k_m'	0		1		2		3		4		5	
α	$C'_{2,\min}$		$C'_{2,\min}$	$Wi_{\max}$	$C'_{2,\min}$	$Wi_{\max}$	$C'_{2,\min}$	$Wi_{\max}$	$C'_{2,\min}$	$Wi_{\max}$	$C'_{2,\min}$	$Wi_{\max}$
0.6	0.033	25.000	0.032	25.833	0.031	26.667	0.030	27.500	0.032	26.200	0.029	29.167
0.7	0.114	6.250	0.103	6.964	0.103	6.950	0.098	7.300	0.093	7.650	0.089	8.000
0.8	0.225	2.778	0.184	3.403	0.175	3.578	0.157	3.978	0.143	4.378	0.131	4.778
0.9	0.356	1.563	0.262	2.118	0.226	2.463	0.191	2.913	0.165	3.363	0.146	3.813
1	0.500	1.000	0.333	1.500	0.250	2.000	0.200	2.500	0.167	3.000	0.143	3.500

Table 2
Variation of p_x' with different values of α, k_m' and constant $Wi = 1$ for a Poiseuille flow with the Giesekus model with the linear Navier slip law.

α	k_m'	1	2	3
0.6		0.666951	0.420398	0.298136
0.7		0.653780	0.419802	0.298037
0.8		X	0.419456	0.298005
0.9		X	0.419394	0.298044
1		X	0.419677	0.298163

can be written using ψ, they always depend on Wi or c_2', so the same approach cannot be used to identify those restrictions.

It was shown that limiting values of Wi and c_2' depend on slip. Increasing the slip coefficient, a higher limiting value for Wi is obtained and the range of admissible solutions for the stress coefficient c_2' also increases, because the admissible values of Wi and c_2' are inversely proportional to each other. Table 1 presents a set of limiting values for Wi and c_2' as a function of α and the slip coefficient.

Although the existence of slip seems to smooth the problem of nonexistence of analytical solutions, such limitation continues to exist, at least for specific cases. In fact, as the Weissenberg number is increased a larger slip velocity is required to guarantee the existence of solution. From Eq. (52) with $Wi = 1, \alpha = 1, k_m' = 0.1$ (linear Navier slip) and some manipulation, the result is, $1/(2c_2') - 1/20 = 1/(1 + \sqrt{1 - 4(c_2')^2})$ an equation without a real number solution, i.e., although slip widens the range of conditions for a solution to exist with the Giesekus model, by itself it does not guarantee its existence.

4.2. Planar channel flow

The symmetry condition of planar Poiseuille flow (see Fig. 1b), defines the shear stress distribution given by $\tau_{xy}^* = Wip_x'y'$ (see Eq. (43a). Using $\phi = -2\alpha Wi(-p_x')$ the differential equation for the velocity derivative (Eq. (43b)) becomes (see [10] for more details),

$$\frac{\partial u^*(y')}{\partial y'} = -\frac{\phi y'\left(1 \pm (2\alpha - 1)\sqrt{1 - \phi^2 y'^2}\right)}{\left(2\alpha - 1 \pm \sqrt{1 - \phi^2 y'^2}\right)^2}. \quad (56)$$

The solution of the direct problem with wall slip is,

$$u^*(y') = \frac{a(\phi)}{\phi} + u_w^*(1) \quad (57)$$

with

$$a(\phi) = (1 - 2b^2)\ln\left[\frac{b + \sqrt{1 - \phi^2 y'^2}}{b + \sqrt{1 - \phi^2}}\right] + b\left(\sqrt{1 - \phi^2 y'^2} - \sqrt{1 - \phi^2}\right) + b(1 - b^2)\left[\frac{1}{b + \sqrt{1 - \phi^2 y'^2}} - \frac{1}{b + \sqrt{1 - \phi^2}}\right]$$

and $b = 2\alpha - 1$.

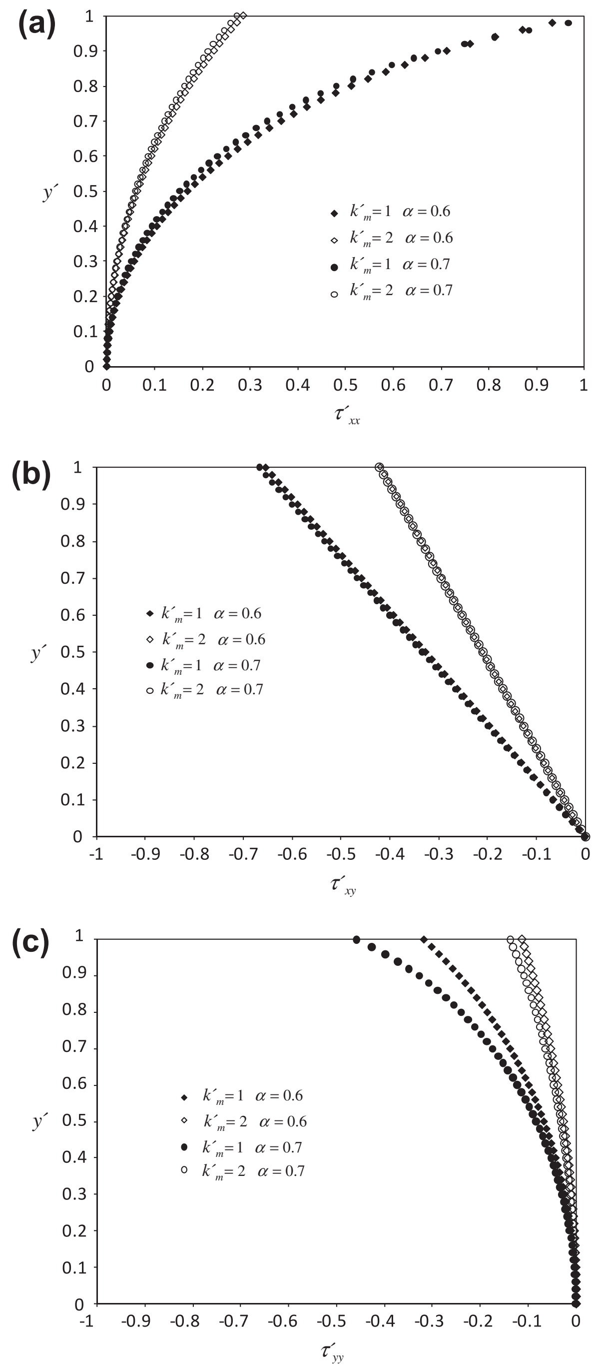

Fig. 5. Variation of τ_{xx}' (a), τ_{xy}' (b) and τ_{yy}' (c) along the channel half width y' for a Poiseuille flow of a Giesekus model with different values of α, k_m' and for a constant $Wi = 1$.

The solution for the inverse problem is obtained as for the PTT fluid, integrating the velocity profile, here with the modified dimensionless velocity

$$\int_0^1 u^*(y')dy' = Wi. \tag{58}$$

Once again the only physically acceptable solution is the upper branch here is given as,

$$\begin{cases} |\phi y'| < 2\alpha\sqrt{1/\alpha - 1} & \text{for } \alpha \in]0; 1/2], \\ |\phi y'| \leqslant 1 & \text{for } \alpha \in]1/2; 1], \end{cases} \tag{59}$$

for a fixed y'. As expected, restrictions on the admissible Weissenberg number Wi and pressure gradient $-p'_x$ arise.

For Poiseuille flow, slip also relaxes the Weissenberg number restriction and in order to obtain the pressure gradient the nonlinear Eq. (58) must be solved numerically.

Assuming that $Wi = 1$ Yoo and Choi [10] showed that there should be no solution for Eq. (58), but its existence can be proved for some cases with slip even though it has to be determined numerically. For this particular case, Table 2 lists the pressure gradient for different values of α and k'_m. No solution exists for $\alpha = 0.8; 0.9; 1.0$ and $k'_m = 1$ and the pressure gradient decreases when α increases.

The variation of the shear stress τ^*_{xy} (and of the other stress components) with slip, shown in Fig. 5, is qualitatively similar to that for the PTT models. For the normal stress τ^*_{xx} the Giesekus model exhibits lower values than the corresponding PTT models, everything else being the same. Even though the effect of α on both τ^*_{xy} and τ^*_{xx} is very small, it leads to a non-zero second normal stress difference (here $N_2 = -\tau^*_{yy}$) that decreases with slip as shown in the plots of τ^*_{yy} of Fig. 5c ($\tau^*_{yy} = 0$ for any of the simplified PTT models).

The solution for the Poiseuille pipe flow of a Giesekus fluid is given in Appendix B.

5. Conclusions

Analytical and semi-analytical solutions (for the direct and inverse problems) are presented for the Couette and Poiseuille flows of linear and exponential simplified PTT fluids, together with an analysis of the existence of solutions for the one mode Giesekus model.

For the sPTT fluids it could be proved that for the four slip models presented there is always a unique solution for the flow between parallel plates, but full analytical solutions could only be found for special values of the exponent in the nonlinear Navier slip law.

For the Giesekus fluid, the procedure to obtain the solution is very similar as the one employed for the sPTT fluid. The proof of existence of solutions (that could not exist without slip velocity) is made analytically for the Couette flow and is studied numerically for the Poiseuille flow. For both flows this study is carried out for the Navier slip law, although, for the other nonlinear laws the results are qualitatively similar.

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Appendix A

A.1. Solutions for the Couette flow and nonlinear Navier slip with the PTT models

The nonlinear Navier slip law is given by the following equation,

$$u' = k'_m (c'_1)^m, \ m \in \mathbb{R}^+, \tag{A1}$$

where $k'_m = kU^{m-1}(\eta/h)^m$. The velocity profile is obtained by solving the following two systems of equations. For the linear PTT model they are,

$$u'(y') = \left[c'_1 + 2\varepsilon Wi^2 (c'_1)^3\right](y') + k'_m (c'_1)^m, \tag{A2}$$

$$2\varepsilon Wi^2 (c'_1)^3 + c'_1 + k'_m (c'_1)^m - 1 = 0. \tag{A3}$$

Let $g(c'_1) = 2\varepsilon Wi^2 (c'_1)^3 + c'_1 + k_m (c'_1)^m - 1$, then, the derivative of $g(c'_1)$ is positive and given by,

$$\frac{dg(c'_1)}{dc'_1} = 6\varepsilon Wi^2 (c'_1)^2 + m \times k'_m (c'_1)^{m-1} + 1 > 0. \tag{A4}$$

Since $g(0) = -1$ and $g(1) = 2\varepsilon Wi^2 + k_m > 0$, Bolzano and Rolle theorems imply a unique solution in the range [0; 1].

For the special case of $m = 3$, the solution is given by Eq. (A5) obtained with the help of the Cardan–Tartaglia formula,

$$\begin{aligned} c'_1 = \Bigg(&\left(2\varepsilon Wi^2 + k'_m\right)^{-1}/2 + \left[\left(\left(-2\varepsilon Wi^2 - k'_m\right)^{-1}/2\right)^2 \right. \\ &\left. + \left(\left(-2\varepsilon Wi^2 + k'_m\right)^{-1}/3\right)^3\right]^{1/2}\Bigg)^{1/3} + \Bigg(\left(2\varepsilon Wi^2 - k'_m\right)^{-1}/2 \\ &- \left[\left(\left(-2\varepsilon Wi^2 - k'_m\right)^{-1}/2\right)^2 + \left(\left(2\varepsilon Wi^2 + k'_m\right)^{-1}/3\right)^3\right]^{1/2}\Bigg)^{1/3}. \end{aligned} \tag{A5}$$

For the special case of $m = 2$, the analytical solution is obtained as a general solution of a cubic equation [24].

The system of equations for the exponential PTT is,

$$u'(y') = \exp\left(2\varepsilon Wi^2 (c'_1)^2\right) c'_1 y' + k'_m (c'_1)^m, \tag{A6}$$

$$\exp\left(2\varepsilon Wi^2 (c'_1)^2\right) c'_1 + k'_m (c'_1)^m - 1 = 0. \tag{A7}$$

Let $g(c'_1) = \exp(2\varepsilon Wi^2 (c'_1)^2) c_1 + k'_m (c'_1)^m - 1$, the derivative of $g(c'_1)$ is positive and given by following equation,

$$\frac{dg(c'_1)}{dc'_1} = \exp\left(2\varepsilon Wi^2 (c'_1)^2\right)\left(1 + 4\varepsilon Wi^2 (c'_1)^2\right) + m \times k'_m (c'_1)^{m-1} > 0 \tag{A8}$$

Since $g(0) = -1$ and $g(k'^{-1/m}_m) = \exp(2\varepsilon Wi^2 (k'_m)^{-2/m}) k'^{-1/m}_m > 0$, we have, once again by Bolzano and Rolle theorems, a unique solution in the interval $[0; k'^{-1/m}_m]$.

For the other two slip boundary conditions given by Eq. (16b) and (16c) we have similar results. For the linear PTT let $g(c'_1) = 2\varepsilon Wi^2 (c'_1)^3 + c'_1 + u'_w(0) - 1$, then the following positive derivative is obtained,

$$\frac{dg(c'_1)}{dc'_1} = 6\varepsilon Wi^2 (c'_1)^2 + \frac{du'_w(0)}{dc'_1} + 1 > 0. \tag{A9}$$

For the exponential PTT let $g(c'_1) = 2\varepsilon Wi^2 (c'_1)^2) c'_1 + u'_w(0) - 1$, then,

$$\frac{dg(c'_1)}{dc'_1} = \exp(2\varepsilon Wi^2 (c'_1)^2)\left(1 + 4\varepsilon Wi^2 (c'_1)^2\right) + \frac{du'_w(0)}{dc'_1} > 0. \tag{A10}$$

Since $g(0) < 0$ and $g(1) > 0$ for both the linear and the exponential PTT once again it is proved the existence of a unique solution in the range [0; 1].

Appendix B. Pipe flow for the sPTT and Giesekus models

B.1. sPTT

The solutions for the pipe flow (Fig. 1c) are very similar to those of channel flow. A practical way to obtain the simplified governing equations is to substitute y' by $r'/2$ in Eq. (15) leading to,

$$\tau'_{rx} = p'_x(r'/2), \tag{B1a}$$

$$\tau'_{xx} = (2\lambda/\eta)(p'_x)^2 r'^{\,2}/4, \tag{B1b}$$

$$\tau'_{rr} = 0, \tag{B1c}$$

$$(\partial u'/\partial r') = f((2\lambda/\eta)(p'_x)^2 r'^{\,2}/4)p'_x(r/n\eta). \tag{B1d}$$

The solution for the direct problem is given by Eqs. (B2) and (B3) for the linear and the exponential models, respectively.

$$u'(r') = 0.125p'_x(r'^2 - 1) + 0.0625\varepsilon Wi^2(p'_x)^3(r'^4 - 1) + u'_w(1) \tag{B2}$$

$$u'(y') = [2\varepsilon Wi^2(p'_x)]^{-1}(\exp(0.5\varepsilon Wi^2(p'_x)^2 r'^2) - \exp(0.5\varepsilon Wi^2(p'_x)^2)) + u'_w(1) \tag{B3}$$

The term $u'_w(1)$ is once again given by any of the Eqs. (17a–c). The solution for the inverse problem is very similar to the channel flow.

B.2. Giesekus

For the pipe flow, the solution is very similar to that in the pressure-driven channel flow. The main difference is that y' is replaced by r', and ϕ gives place to $\Psi = \alpha Wi(-p'_x)$.

$$u^*(r') = \frac{a(\psi_1)}{\psi_1} + u^*_w(1). \tag{B4}$$

Appendix C. Supplementary material

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jnnfm.2012.01.009.

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