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Short communication

A generalized Brinkman number for non-Newtonian duct flows

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ABSTRACT

When viscous dissipation effects are important in duct flows the Brinkman number is widely used to quantify the relationship between the heat generated by dissipation and the heat exchanged at the wall. For Newtonian laminar fully developed pipe flow the use of the classical expression for this dimensionless group is appropriate, but under different conditions it can lead to misleading conclusions, such as when comparing flows through different cross-section ducts, flow regimes and mainly non-Newtonian flows. In this work a generalized Brinkman number is proposed, based on an energy balance for the power dissipated by friction, that allows proper quantification of viscous heating effects and reduces to the classical definition in laminar Newtonian pipe flow. The advantages of the new definition are shown and expressions are given for generalized Brinkman numbers in the most common cases.

1. Introduction

When dissipation effects are important in fluid mechanics the Brinkman number is widely used to quantify the relationship between the heat generated by dissipation and the heat exchanged at the wall. This dimensionless group was named after Brinkman, who solved the Newtonian Poiseuille pipe flow problem with viscous dissipation [1].

For Newtonian laminar pipe flow the use of the classical definition of the Brinkman number is straightforward and provides an adequate estimate of the ratio between the heat generated by viscous heating and the heat exchanged at the pipe wall. However, outside this flow condition [2–7], the same definition quantifies different values of that ratio leading to an incorrect interpretation of some results. A simple example is the comparison between laminar and turbulent pipe flows at the same Reynolds number of around 3000 and equal wall heat transfer. The classical Brinkman number has a unique value, but the friction heat in turbulent flow is clearly larger than in laminar flow. This difference is captured by the generalized definition of the Brinkman number. Other examples of more interest to this audience are comparisons of duct flows of Newtonian and non-Newtonian fluids in the same or in different cross-section ducts and especially including fluids of variable viscosity. The classical Brinkman number restricts any attempt to compare dissipation effects between different fluids or flow regimes even flowing in the same geometry. The emergence of micro and nanotechnology applications, where surface forces gain relevance, emphasizes the role of viscous dissipation in geometries that are seldom circular [8].

The definition of a general Brinkman number is the objective of this work and is aimed at a comprehensive and correct quantification of dissipation effects, regardless of the rheological constitutive equation adopted and of flow regime, thus facilitating meaningful comparisons of viscous effects over a wide variety of Newtonian and non-Newtonian, laminar and turbulent duct flows.

This paper starts with the general definition of Brinkman number and then derives the classical Brinkman number as a particular case of the former. Section 3 justifies the unified definition of Brinkman number showing that it leads to a more correct interpretation of several examples and that its use is only marginally more expensive than the classical definition.

2. The generalized Brinkman number

A unified definition of Brinkman number compares the energy dissipated internally as heat and the flux of thermal energy transferred at the wall, denoted $\dot{q}_{\mathrm{w}}$. An energy balance on a duct segment of constant cross-section, provides the following expression for the power dissipated by friction [9], $\dot{W}$,

$$\dot{W} = \Delta p \dot{V} \tag{1}$$

where Δp is the frictional pressure drop and $\dot{V}$ is the volume flow rate. For a duct of infinitesimal length, dx, the power per unit wall area dissipated by friction, $\dot{w}$, is related to the frictional pressure

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gradient by

$$\dot{w} = -\frac{\mathrm{d}p}{\mathrm{d}x}\frac{A}{P}\bar{U} = -\frac{\mathrm{d}p}{\mathrm{d}x}\frac{D_\mathrm{h}}{4}\bar{U} \tag{2}$$

where A, P and D_h are the duct cross-section area, duct-wetted perimeter and hydraulic diameter, respectively, $\bar{U}$ is the flow bulk velocity and the minus sign ensures positive work. This expression is general and considers both fully developed flow (constant pressure gradient) as well as developing flow (variable pressure gradient). In addition, this definition can be used to quantify viscous dissipation in other cases such as flow through porous media, as shown later.

For fully developed flow the momentum balance relates the pressure variation with the wall shear stress, τ_w, by $\tau_\mathrm{w} = -(\mathrm{d}p/\mathrm{d}x)D_\mathrm{h}/4$ so that the power dissipated by friction can be expressed as a function of the wall shear stress by $\dot{w} = \tau_\mathrm{w}\bar{U}$.

The generalized form of the Brinkman number, henceforth denoted by Br^*, would be expressed uniquely as the ratio $\dot{w}/\dot{q}_\mathrm{w}$, but a slight modification is introduced to ensure that it reduces to the classical definitions of the Brinkman number for Newtonian laminar pipe flow. This introduces a coefficient 8 and the generalized Brinkman number is defined as

$$Br^* = \frac{\dot{w}}{8\dot{q}_\mathrm{w}}. \tag{3}$$

Note that for generalized Brinkman number to arise naturally when normalizing the energy conservation equation it suffices to make the stress tensor nondimensional with the wall shear stress.

Substituting $\dot{w}$, Br^* assumes the particular forms of Eqs. (4a) and (4b) for constant wall heat flux and constant wall temperature, respectively.

$$Br^* = \frac{\bar{U}\tau_\mathrm{w}}{8\dot{q}_\mathrm{w}} \tag{4a}$$

$$Br^* = \frac{\bar{U}\tau_\mathrm{w}D}{8k(T_\mathrm{w} - T_0)}, \tag{4b}$$

here T_w denotes the wall temperature, T_0 is the inlet bulk temperature, k is the fluid thermal conductivity and D is the pipe diameter. As mentioned, the coefficient 8 is introduced for consistency with previous definitions.

For fully developed duct flows the wall shear tress is related to the Darcy friction factor, f, by the definition of Eq. (5) where ρ is the fluid density

$$\tau_\mathrm{w} = \frac{f\bar{U}^2\rho}{8}. \tag{5}$$

For laminar pipe flow $f = 64/Re$, where Re is the Reynolds number and by back substitution the classical Brinkman numbers are recovered. These are Eqs. (6a) and (6b) for the constant wall heat flux and the constant wall temperature, respectively, with η here denoting the dynamic viscosity of the Newtonian fluid.

$$Br^* = Br \equiv \frac{\eta\bar{U}^2}{\dot{q}_\mathrm{w}D} \tag{6a}$$

$$Br^* = Br \equiv \frac{\eta\bar{U}^2}{k(T_\mathrm{w} - T_0)}. \tag{6b}$$

The generalized Brinkman number provides a unique and physically based definition that incorporates dissipation effects regardless of rheological constitutive model, duct size, duct shape and flow regime valid for both the laminar and turbulent regimes, including flows through porous media. It remains to be seen whether the use of Br^* is as simple as the classical definition, an issue addressed in the next section.

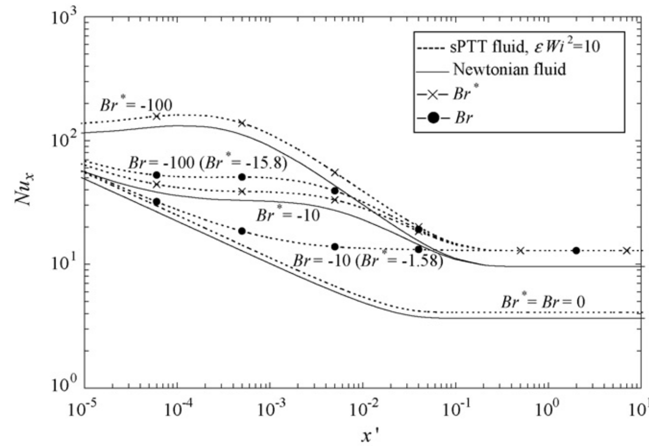

Fig. 1. Variation of Nusselt number in a thermally developing pipe flow for fixed wall temperature with cooling at Brinkman numbers of 0, −10 and −100. Solid line—Newtonian fluid; dashed line—sPTT fluid ($\varepsilon Wi^2 = 10$). Symbols: × generalized Brinkman number, Br^*; ● classical Brinkman number, Br.

3. Examples of application

3.1. Fully developed laminar duct flows of non-Newtonian fluids

For fully developed laminar duct flows of non-Newtonian fluids with a characteristic shear viscosity η the wall shear stress can be rewritten as a function of the bulk velocity of a Newtonian fluid flowing under the same wall shear stress, $\bar{U}_\mathrm{N}$ as

$$\tau_\mathrm{w} = \frac{fRe\eta\bar{U}_\mathrm{N}}{8D_\mathrm{h}}. \tag{7}$$

The product fRe, here applied to Newtonian fluids is constant for a given duct cross-section, [9]. Back substitution into Eq. (4a) or (4b) the following relation between Br^* and Br in laminar flow is derived

$$Br^* = \frac{fRe}{64}Br\frac{\bar{U}_\mathrm{N}}{\bar{U}}. \tag{8}$$

The advantage of the generalized Brinkman number stands out clearly when comparing results plotted as a function of the classical and generalized Brinkman numbers. This is done next for the Graetz problem in a pipe flow for Newtonian and viscoelastic fluids represented by the simplified Phan–Thien–Tanner (sPTT) fluid, originally investigated by Coelho et al. [10].

This flow has a nondimensional number measuring the fluid flow elasticity of $\varepsilon Wi^2 = 10$, where ε is a constitutive parameter and Wi is the Weissenberg number. Of relevance are the variations of the local Nusselt number along the thermal entrance region of the pipe shown in Fig. 1. This figure compares data for different numerical values of the Brinkman number using both the classical (Br, see [10] for details on dimensionless numbers) and the generalized definitions (Br^*). The cases plotted pertain to imposed wall temperatures, so the relevant definitions are those involving $k(T_\mathrm{w} - T_0)/D$ rather than $\dot{q}_\mathrm{w}$. The Brinkman numbers are negative, because the inlet temperature is higher than the wall temperature (wall cooling).

In the absence of viscous dissipation, the Nusselt number for the sPTT fluids is higher than for the Newtonian fluid by a small amount, but the two curves are essentially parallel. For Newtonian fluids there is a significant increase in the heat transfer coefficient with increasing viscous dissipation effects, especially in the entrance region, but for fully developed flow the Nusselt number is independent of the Brinkman number. Note that for Newtonian laminar pipe flow the generalized and classical Brinkman numbers are identical, cf. Eqs. (6a), (6b) and (8).

Table 1
Generalized Brinkman number in laminar flow of an sPTT fluid

(a) Pipe flow		
τ_w	$\bar{U}_N\bar{U}(\text{with } \bar{U}_N \equiv [-(dp/dx)R^2]/(8\eta))$	$Br^* = Br\dfrac{\bar{U}_N}{\bar{U}}$
$8\eta\dfrac{\bar{U}}{D}\dfrac{\bar{U}_N}{\bar{U}}$	$\dfrac{\left[\sqrt{144\varepsilon Wi^2+1}+12\sqrt{\varepsilon Wi^2}\right]^{1/3}-\left[\sqrt{144\varepsilon Wi^2+1}-12\sqrt{\varepsilon Wi^2}\right]^{1/3}}{8\sqrt{\varepsilon Wi^2}}$	$\dfrac{\eta\bar{U}^2}{\dot{q}_w D}\dfrac{\bar{U}_N}{\bar{U}}$
(b) Parallel plates flow		
τ_w	$\bar{U}_N/\bar{U}(\text{with } \bar{U}_N \equiv [-(dp/dx)R^2/(3\eta)])$	$Br^* = \dfrac{3}{2}Br\dfrac{\bar{U}_N}{\bar{U}}$
$12\eta\dfrac{\bar{U}}{D_h}\dfrac{\bar{U}_N}{\bar{U}}$	$\dfrac{10^{1/3}\left\{\left[\sqrt{729\varepsilon Wi^2+10}+27\sqrt{\varepsilon Wi^2}\right]^{1/3}-\left[\sqrt{729\varepsilon Wi^2+10}-27\sqrt{\varepsilon Wi^2}\right]^{1/3}\right\}}{18\sqrt{\varepsilon Wi^2}}$	$\dfrac{3}{2}\dfrac{\eta\bar{U}^2}{\dot{q}_w D_h}\dfrac{\bar{U}_N}{\bar{U}}$
(c) Annular flow		
$\bar{\tau}_w$	$\bar{U}_C/\bar{U}(\text{with } \bar{U}_C = (-dp/dx)D_h^2/32\eta)$	$Br^* = Br\dfrac{\bar{U}_C}{\bar{U}}$
$8\eta\dfrac{\bar{U}}{D_h}\dfrac{\bar{U}_C}{\bar{U}}$	Eq. (16) [12] for sPTT fluids and Eq. (6) [13] for Newtonian fluids	$\dfrac{\eta\bar{U}^2}{\bar{\dot{q}}_w D_h}\dfrac{\bar{U}_C}{\bar{U}}$

η—Polymer viscosity coefficient in sPTT model. R—Pipe radius or channel half height, $Wi \equiv \lambda\bar{U}/R$.

Inspecting next the sPTT plots for $Br \neq 0$ even though the Nusselt number has increased significantly in comparison with the $Br = 0$ case, their curves now fall below those for the Newtonian fluid at the same value of the classical Brinkman number and the corresponding sPTT and Newtonian curves cease to be parallel, even though the velocity profiles remain unchanged. However, using the generalized Brinkman number moves the *Nu* curve for the sPTT fluid to a place slightly above and parallel to the Newtonian curve as previously seen in the absence of viscous dissipation. Regardless of the value of the generalized Brinkman number, at the same value of Br^* the curves pertaining to the sPTT and Newtonian fluids maintain the same shape and relative difference, because the velocity profiles and the ratio $\dot{w}/\dot{q}_w$ for each fluid are the same leading to a coherent set of results. This was not the case when using the classical Brinkman number definition, which is related to the viscous dissipation in a inconsistent manner. Note that $Br = -100$ and $Br = -10$ correspond to $Br^* = -15.8$ and $Br^* = -1.58$, respectively.

There is a wealth of analytical solutions in the literature for heat transfer in laminar duct flows of fluids represented by the simplified Phan–Thien—Tanner and FENE-P models, accounting for viscous dissipation effects, all of which are presented in terms of the classical Brinkman numbers [10,11]. Our recommendation, which is equally valid for fluids described by other rheological constitutive equation, is to reconsider those results using instead the general-

Table 2
Generalized Brinkman number in laminar pipe and channel flows of a Bingham fluid

(a) Pipe flow		
τ_w	$\bar{U}_N/\bar{U}(\text{with } \bar{U}_N \equiv [-(dp/dx)R^2]/(8\mu_0))$	$Br^* = Br\dfrac{\bar{U}_N}{\bar{U}}$
$8\mu_0\dfrac{\bar{U}}{D}\dfrac{\bar{U}_N}{\bar{U}}$	$\left[1-\dfrac{4}{3}\dfrac{\tau_0}{\tau_w}+\dfrac{1}{3}\left(\dfrac{\tau_0}{\tau_w}\right)^4\right]^{-1}$	$\dfrac{\mu_0\bar{U}^2}{\dot{q}_w D}\dfrac{\bar{U}_N}{\bar{U}}$
The ratio τ_0/τ_w is obtained by solving the following equation $\dfrac{\bar{U}}{D\tau_0/(8\mu_0)} = \left(\dfrac{\tau_0}{\tau_w}\right)^{-1}\left[1-\dfrac{4}{3}\dfrac{\tau_0}{\tau_w}+\dfrac{1}{3}\left(\dfrac{\tau_0}{\tau_w}\right)^4\right]$ and the solution is $\dfrac{\tau_0}{\tau_w} = \dfrac{2^{5/6}}{4}\left\{\sqrt{(B^{1/3}+C^{1/3})}-\sqrt{\left[2\sqrt{(B^{2/3}-B^{1/3}C^{1/3}+C^{2/3})}-B^{1/3}-C^{1/3}\right]}\right\}$ with $B = (3U^+ +4)^2-\sqrt{3U^+(27U^{+3}+144U^{+2}+288U^+ +256)}$, $C = (3U^+ +4)^2+\sqrt{3U^+(27U^{+3}+144U^{+2}+288U^+ +256)}$ and $U^+ = \dfrac{\bar{U}}{D\tau_0/(8\mu_0)}$		
(b) Parallel plates flow		
τ_w	$\bar{U}_N/\bar{U}\,(\text{with } \bar{U}_N \equiv [-(dp/dx)R^2]/(3\mu_0))$	$Br^* = \dfrac{3}{2}Br\dfrac{\bar{U}_N}{\bar{U}}$
$12\mu_0\dfrac{\bar{U}}{D_h}\dfrac{\bar{U}_N}{\bar{U}}$	$\left[1-\dfrac{3}{2}\dfrac{\tau_0}{\tau_w}+\dfrac{1}{2}\left(\dfrac{\tau_0}{\tau_w}\right)^3\right]^{-1}$	$\dfrac{3}{2}\dfrac{\mu_0\bar{U}^2}{\dot{q}_w D_h}\dfrac{\bar{U}_N}{\bar{U}}$
The ratio τ_0/τ_w is obtained by solving the following equation $\dfrac{\bar{U}}{D_h\tau_0/(12\mu_0)} = \left(\dfrac{\tau_0}{\tau_w}\right)^{-1}\left[1-\dfrac{3}{2}\dfrac{\tau_0}{\tau_w}+\dfrac{1}{2}\left(\dfrac{\tau_0}{\tau_w}\right)^3\right]$ and the solution is $\dfrac{\tau_0}{\tau_w} = \dfrac{2}{3}\sqrt{3(2U^+ +3)}\sin\left\{\dfrac{1}{3}\arctan\left[\dfrac{3\sqrt{6}}{2\sqrt{U^+(4U^{+2}+18U^+ +27)}}\right]\right\}$ with $U^+ = \dfrac{\bar{U}}{D_h\tau_0/(12\mu_0)}$		

μ_0 and τ_0 are the Bingham model rheological parameters, $\eta = \mu_0 + \tau_0/\dot{\gamma}, \quad \tau \geq \tau_0$ and $\dot{\gamma} = 0, \quad \tau \leq \tau_0$. R—Pipe radius or the channel half height.

Table 3
Coefficients a and b of Eq. (10) for the square duct, parallel plates and pipe

Square duct	
a = 0.2121	b = 0.6766
Parallel plate	
a = 0.5000	b = 1.0000
Pipe	
a = 0.2500	b = 0.7500

ized Brinkman number Br^*. This implies some transformations for existing solutions in the literature: for sPTT and Bingham fluids for example, it suffices to substitute Br by $Br^*(64/fRe)(\bar{U}/\bar{U}_N)$, (cf. Eq. (8)). This modification is easy to implement, because usually the existing solution already contains the product $Br(\bar{U}_N/\bar{U})$ in an explicit or implicit form. In comparisons where only the fluids differ it is also acceptable to drop $64/fRe$ although the resulting Brinkman number is no longer the original generalized Brinkman number of Section 2.

The ratio $\bar{U}_N/\bar{U}$ assumes different forms for different duct sections and fluids. Examples are summarized in Tables 1 and 2 for the sPTT and Bingham fluids, respectively. Note that the expressions for $\bar{U}_N/\bar{U}$ in Table 1 for the pipe and channel flows are simpler to use than the general equation given in [12]. For the annular flow the perimeter averaged wall shear stress, $\bar{\tau}_w$, and wall heat flux, $\bar{\dot{q}}_w$, should be used in Br^*.

For the sake of simplicity, in the remaining part of the paper results will be presented only for the constant wall heat flux case.

For fully developed duct flows of power-law fluids, an expression for the generalized Brinkman number can be deduced using Eq. (5) and the corresponding fRe expression which is

$$fRe = f\frac{\rho\bar{U}^{2-n}D_h^n}{K} = 2^{3n+3}\left(\frac{a+bn}{n}\right)^n, \tag{9}$$

derived after the work of Kozicki et al. [14], who provided tables for parameters a and b as a function of the shape of the duct cross-section. These are reproduced in Table 3 for the square duct, the parallel plates and pipe.

Back-substituting in Eqs. (5) and (4a) leads to Eq. (10) for the generalized Brinkman number

$$Br^* = 2^{3n-3}\left(\frac{a+bn}{n}\right)^n\frac{K\bar{U}^{1+n}}{\dot{q}_w D_h^n}. \tag{10}$$

In his investigation Barletta [15] defined the Brinkman number as $Br = K\bar{U}^{n+1}/\dot{q}_w D^n$, but in his Eq. (40) Br appears multiplied by $2^{n-3}[(3n+1)/n]^n$ giving precisely the generalized Brinkman number in Eq. (10) with a = 0.25 and b = 0.75, cf. Table 3.

For n = 1 Eq. (10) gives Br^* = 0.889Br and Br^* = 1.5Br for flows in square ducts and between parallel plates, respectively. These expressions put in evidence the importance of using the generalized Brinkman number when comparing dissipation effects between different duct flows. While Br indicates the same level of viscous dissipation Br^* reveals an increase of 70% for a flow between parallel plates relative to the square duct flow, i.e., the use of Br^* is more enlightening than the use of Br. For Newtonian fluids the classical Brinkman number approach is only adequate if we are dealing with the same duct cross-section and flow regime.

Finally, a change in flow regime is also dealt with correctly by Br^*. Whereas with the classical Brinkman number there is no difference in the numerical value of Br for flows in the laminar and turbulent regimes, all other conditions being identical, and in particular the Reynolds number, the corresponding numerical values of Br^* will differ to express the fact that viscous heating is higher in turbulent flow than in laminar flow. For instance, at a Reynolds number of around 3000 the value of Br^* under turbulent flow conditions is about two times larger than the corresponding laminar value expressing the amount of heat generated internally by friction as accounted for by the presence of the wall shear stress in the definition of Br^*.

3.2. Flow through porous media

The usual Brinkman number Br for a Newtonian flow through porous media is defined in Eq. (6a), whereas the generalized Brinkman number is given by Eq. (11) obtained after substituting Eq. (2) into Eq. (3). Incidentally, in the context of injection moulding of plastics Janeschitz–Kriegl [16] used a similar equation.

$$Br^* = \frac{-(dp/dx)(A/P)\bar{U}}{8\dot{q}_w}. \tag{11}$$

The frictional pressure drop and the bulk velocity for flow through a porous medium are related by Darcy's equation [2], $\bar{U} = -(dp/dx)k/\eta$, where κ is its permeability. Back substituting into Eq. (11) gives the following relation between Br^* and Br

$$Br^* = \frac{Br}{32\,\mathrm{Da}}, \tag{12}$$

where Da is the Darcy number defined as κ/D_h^2. Incidentally, for flow through a saturated porous medium the relevant parameter is Br/Da rather than Br, as reported by Nield [17], i.e., proportional to the generalized Brinkman number (cf. Eq. (12)).

4. Conclusions

The generalized Brinkman number is a unified definition valid for any fluid, duct and/or flow regime correctly quantifying the ratio between frictional heat and heat transfer at the wall, thus clarifying the physics of flow phenomena. Its use is only marginally costlier than that of the classical Brinkman number, especially because the formal expression for Br^* appears often disguised in published solutions with the failure to recognize its presence leading to some misinterpretations of results.

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References

[1] H.C. Brinkman, Heat effects in capillary flow I, Appl. Sci. Res. A2 (1951) 120–124.
[2] R.B. Bird, W.E. Stewart, E.N. Lightfoot, Transport Phenomena, 2nd ed., John Wiley, New Delhi, 2006, pp. 300–355.
[3] M. Avcı, O. Aydın, Laminar forced convection slip-flow in a micro-annulus between two concentric cylinders, Int. J. Heat Mass Transfer 51 (2008) 3460–3467.
[4] R. Khatyr, D. Ouldhadda, A. Il Idrissi, Viscous dissipation effects on the asymptotic behaviour of laminar forced convection for Bingham plastics in circular ducts, Int. J. Heat Mass Transfer 46 (2003) 589–598.
[5] G. Tunc, Y. Bayazitoglu, Heat transfer in microtubes with viscous dissipation, Int. J. Heat Mass Transfer 44 (2001) 2395–2403.
[6] K.J. Hammad, The effect of hydrodynamic conditions on heat transfer in a complex viscoplastic flow field, Int. J. Heat Mass Transfer 43 (2000) 945–962.
[7] T. Min, J.Y. Yoo, H. Choi, Laminar convective heat transfer of a Bingham plastic in a circular pipe. I. Analytical approach-thermally fully developed flow and thermally developing flow (the Graetz problem extended), Int. J. Heat Mass Transfer 40 (1997) 3025–3037.
[8] H. Bruus, Theoretical Microfluidics, Oxford University Press, Oxford, 2008, p. 41.
[9] F.M. White, Fluid Mechanics, 4th ed., McGraw-Hill, Boston, 1999, pp. 168–365.
[10] P.M. Coelho, F.T. Pinho, P.J. Oliveira, Thermal entry flow for a viscoelastic fluid: the Graetz problem for the PTT model, Int. J. Heat Mass Transfer 46 (2003) 3865–3880.

[11] F.T. Pinho, P.M. Coelho, Fully-developed heat transfer in annuli for viscoelastic fluids with viscous dissipation, J. Non-Newt. Fluid Mech. 138 (2006) 7–21.
[12] F.T. Pinho, P.J. Oliveira, Axial annular flow of a nonlinear viscoelastic fluid—an analytical solution, J. Non-Newt. Fluid Mech. 93 (2000) 325–337.
[13] P.M. Coelho, F.T. Pinho, Fully-developed heat transfer in annuli with viscous dissipation, Int. J. Heat Mass Transfer 49 (2006) 3349–3359.
[14] W. Kozicki, C.H. Chou, C. Tiu, Non-Newtonian flow in ducts of arbitrary cross-sectional shape, Chem. Eng. Sci. 21 (1966) 665–679.
[15] A. Barletta, Fully developed laminar forced convection in circular ducts for power-law fluids with viscous dissipation, Int. J. Heat Mass Transfer 40 (1997) 15–26.
[16] H. Janeschitz-Kriegl, Injection moulding of plastics. II. Analytical solution of heat transfer problem, Rheol. Acta 18 (1972) 693–701.
[17] D.A. Nield, The modeling of viscous dissipation in saturated porous medium, ASME J. Heat Transfer 129 (2007) 1459–1463.