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The Graetz problem with viscous dissipation for FENE-P fluids

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Abstract

The solution of thermal entry problem for pipe and channel flow of highly viscous liquids having viscoelastic properties, here modeled by the FENE-P constitutive equation, is obtained semi-analytically. Two types of boundary condition are considered, either an imposed wall temperature or a prescribed wall heat flux, and the effects of viscous dissipation are also taken into account. The analysis leads to a Sturm-Liouville problem similar to that found for the simplified Phan-Thien–Tanner fluid with linear stress coefficient, provided an adequate transformation of constitutive variables is performed first. The way to do this is explained and representative results for thermal entry flow of the FENE-P fluid are given.

1. Introduction

In polymer processing, highly elastic fluids flow under non-isothermal conditions, but such flows may usually be considered as dynamically fully developed, because of the high viscosities involved. Hence, the thermal entry problem in channels and pipes is very frequent and the issue of viscous dissipation is also quite relevant due to the combination of large viscosities and shear rates.

There are very few investigations of the thermal entry flow for viscoelastic fluids; these were recently reviewed by Coelho et al. [1], who derived analytically a solution for the Phan-Thien–Tanner (PTT) model in its simplified version with a linear stress function, under various boundary conditions. Previous investigations by the same group had already provided asymptotic heat transfer solutions in pipe and channel flows for the same fluids [2,3].

Another very common rheological constitutive equation for polymeric liquids is the non-linear dumbbell model, such as the FENE-P equation, which was derived for dilute solutions but may be extended to semi-dilute and concentrate solutions following the ideas of the encapsulated dumbbell model [4]. The fully developed dynamical isothermal solution for the FENE-P model has been derived by Oliveira [5], but the literature is very scarce regarding its performance in heat transfer problems.

This paper presents a solution of the thermal entry flow in pipes and channels for the FENE-P fluid, for imposed constant wall temperature and wall heat flux, in the presence of viscous dissipation. Although more often multimode constitutive equations are being used, single mode versions can be sufficiently accurate in pure shear flows, as in here.

2. Governing equations and method of solution

The entry flow problem of very viscous fluids is characterized by fully developed dynamics and a thermally developing flow. Since the model parameters are assumed to be independent of temperature, the dynamical problem is decoupled from the thermal problem. The duct is aligned in the x-direction with the origin at $x = 0$ and is either a pipe (index $n = 1$) or a channel ($n = 0$) and two types of wall boundary conditions are considered: (i) constant wall

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temperature T_w; and (ii) constant wall heat flux $\dot{q}_w$. At $x = 0$ the thermal boundary condition is applied and for $x < 0$, $T = T_0$, the bulk temperature at inlet. The radial or lateral coordinate is denoted r, with R meaning either the pipe radius or the channel half-height.

The temperature profile is the solution of the following energy conservation equation:

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k \frac{\partial T}{\partial r}\right) + \tau_{xr}\frac{\mathrm{d}u}{\mathrm{d}r}, \tag{1}$$

subject to adequate thermal boundary conditions. Therefore, the solution requires prior knowledge of the velocity and shear stress profiles.

Oliveira [5] has obtained the fully developed axial velocity profile (u) for the FENE-P model, which is given in non-dimensional form by

$$\frac{u}{U} = \frac{3+n}{2}\frac{U_N}{U}\left[1-\left(\frac{r}{R}\right)^2\right]\left\{1+\beta\left[1+\left(\frac{r}{R}\right)^2\right]\right\}, \tag{2}$$

where

$$\beta = (3+n)^2 \frac{We^2}{a^2L^2}\left(\frac{U_N}{U}\right)^2, \tag{3}$$

and the shear stress profile is

$$\frac{\tau_{xr}}{\eta U/R} = -(n+3)\frac{U_N}{U}\frac{r}{R}, \tag{4}$$

In Eqs. (2)–(4), U and U_N stand for the cross-section average velocity of the FENE-P fluid and of a Newtonian fluid under the same pressure gradient (i.e., $U_N = -R^2(\mathrm{d}p/\mathrm{d}x)/(2^n(3+n)\eta)$), respectively. We is the Weissenberg number ($We \equiv \lambda U/R$), and a, L^2 and η are constitutive parameters, explained below.

The non-linear differential rheological constitutive equation gives the stress tensor ($\boldsymbol{\tau}$) as a function of a configuration tensor ($\mathbf{A}$), as follows.

$$\boldsymbol{\tau} = \frac{\eta}{\lambda}(f\mathbf{A} - a\mathbf{I}), \tag{5}$$

$$\overset{\nabla}{\mathbf{A}} = -\frac{1}{\lambda}(f\mathbf{A} - a\mathbf{I}), \tag{6}$$

Above, η stands for the zero-shear rate polymer viscosity, λ is the relaxation time and f is the spring force function given by a Warner expression which, after introducing Peterlin's approximation, may be written as

$$f \equiv f(\mathrm{tr}\mathbf{A}) = \frac{L^2}{L^2 - \mathrm{tr}\mathbf{A}}, \tag{7}$$

L^2 is the extensibility parameter that also appears via a, which is defined by

$$a \equiv \frac{1}{1-3/L^2}, \tag{8}$$

i.e., there are three independent constitutive parameters: η, λ and L^2.

The interesting finding regarding the dynamic solution (Eqs. (2) and (3)) is its similarity with the dynamical solution for a simplified PTT fluid with a linear stress coefficient, derived by Oliveira and Pinho [6]. The solution is also given by Eq. (2) but with β defined differently as

$$\beta = (3+n)^2 \varepsilon We^2\left(\frac{U_N}{U}\right)^2, \tag{9}$$

where ε is the elongational parameter of the PTT model. From equality of Eqs. (3) and (9) we can see that

$$\varepsilon = \frac{1}{a^2L^2} = \frac{\left(1-3/L^2\right)^2}{L^2}, \tag{10}$$

Note also that the cubic equations giving U_N/U for the FENE-P and PTT models are identical, provided this transformation is carried out.

The shear stress profiles for the linear PTT and the FENE-P fluids are a result of identical lateral momentum equations.

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \boldsymbol{\tau}_{xr}\right), \tag{11}$$

and can also be matched using Eq. (10).

Thus, we end up with the same energy equation, Eq. (1) to be solved for the temperature distribution $T(r,x)$, using the following boundary conditions:

$$\text{At inlet}: T(r, x=0) = T_0, \tag{12}$$

$$\text{At the symmetry axis}: \frac{\partial T(r=0,x)}{\partial r} = 0, \tag{13}$$

At the wall ($r = R$)

$$\text{either } T(r,x) = T_w, \text{ or } k\frac{\partial T(r,x)}{\partial r} = \dot{q}_w, \tag{14}$$

When the energy equation is expressed in dimensionless form there arises the Brinkman number, a measure of viscous dissipation effects. It is defined as $Br = \eta U^2/[k(T_w - T_0)]$, for the given wall temperature case, or $Br = \eta U^2/[q_w 2R(2-n)]$, for the given wall heat flux case.

Therefore, the main conclusion is that the solution of the thermal entry flow for the FENE-P fluid is exactly the same as that for the simplified PTT model with linear stress coefficient given by Coelho et al. [1], provided that the substitution embodied in Eq. (10) is made. Note here the use of β to represent what in [1] was denoted by a, because a now has a special meaning in the context of the FENE-P equation (see Eq. (8) above).

Before proceeding, it is important to realize that the main differences between the FENE-P model and the simplified PTT model with linear stress coefficient lie on their differing transient responses in shear and extensional flows; as seen above, the steady state responses can be made to collapse with appropriate algebraic transformations. Since the fluids under analysis are usually very viscous, the dynamic flow develops very quickly compared with the thermal flow and, consequently, the thermal entry flow is indeed a steady state flow from a rheological point of view, hence the thermal solutions for the two models are basically ruled by the same final equations with appropriate choice of parameter β (Eqs. (3) or (9)).

The solution to this entry flow problem, where the differential equation on $T(r, x)$ is to be solved, is based on the separation solution methods described in Mikhailov and Özisik [7] and leads to an eigenvalue problem. The determination of the eigenvalues, and of the corresponding eigenfunctions, was accomplished by means of the freeware Fortran code SLEDGE (Pruess and Fulton, Netlib, cited by Pryce [8]), which was adequately modified. Full details of the solution procedure followed are to be found in Coelho et al. [1] and the codes are available from the internet (at http://www.dem.uminho.pt/people/ftp/research/sturmptt.html).

3. Representative results and discussion

It is relevant to present here results for at least one case in order to judge, in a quantitative way, the effects of molecular extensibility and viscous dissipation. The case of pipe heating is selected, which corresponds to a prescribed positive wall heat flux, $\dot{q}_w > 0$, and thus $Br > 0$ (see definition above).

In Fig. 1, the longitudinal variation of the Nusselt number is plotted as a function of We and L^2 for negligible viscous dissipation. The Nusselt number increases with fluid elasticity, but variations are small, not exceeding 20% under thermally fully developed conditions (large x'): for Newtonian fluids $\mathrm{Nu_{fd}} = 4.364$ whereas for $We = 10$ and $L^2 = 10$, $\mathrm{Nu_{fd}} = 4.908$. At very high elasticity, for instance when $We^2/(aL)^2 = 100$, $\mathrm{Nu_{fd}} = 5.053$. The normalised coordinate is defined as $x' \equiv x/(D_H Pe)$, where D_H is the hydraulic diameter ($D_H \equiv 2(2-n)R$) and Pe is the Péclet number, $Pe = UD_H/\alpha$ (α being the thermal diffusivity $\alpha \equiv k/(\rho c_p)$. In addition, for the imposed wall heat flux case the Nusselt number is defined by

$$Nu = \frac{hD_H}{k} = \frac{\dot{q}_w D_H}{k\,(T_w - T_b)}, \tag{15}$$

with h and k representing the heat transfer coefficient and thermal conductivity, respectively.

In Fig. 2, the combined effects of viscous dissipation and fluid elasticity are assessed. Viscous dissipation decreases the Nusselt number, because the fluid closer to the wall gets warmer, i.e. the wall temperature increases faster than the bulk temperature of the fluid so that, for the same wall heat flux the heat transfer coefficient need not be as high. In contrast, fluid elasticity, measured by the Weissenberg number, increases the Nusselt number because of the distortion in the velocity profile: as We increases, shear-thinning intensifies and the velocity profile becomes flatter, leading to higher shear rates in the wall region and improved wall heat transfer, thus reducing the temperature variation across the duct and increasing the Nusselt number. This effect is further intensified by lowering the extensibility parameter, L^2.

The difference between the behaviours of the PTT and FENE-P fluids in this pure shear flow lies on the effects of ε and L^2. Whereas in the former an increase in both ε and We made the velocity profile flatter and thus increased the Nusselt number, for the latter L^2 and We have opposite effects with We increasing shear-thinning and the Nusselt number and higher values of L^2 decreasing shear-thinning and Nu (in fact, Eq. (10) yields $\varepsilon \approx 1/L^2$ for large L^2).

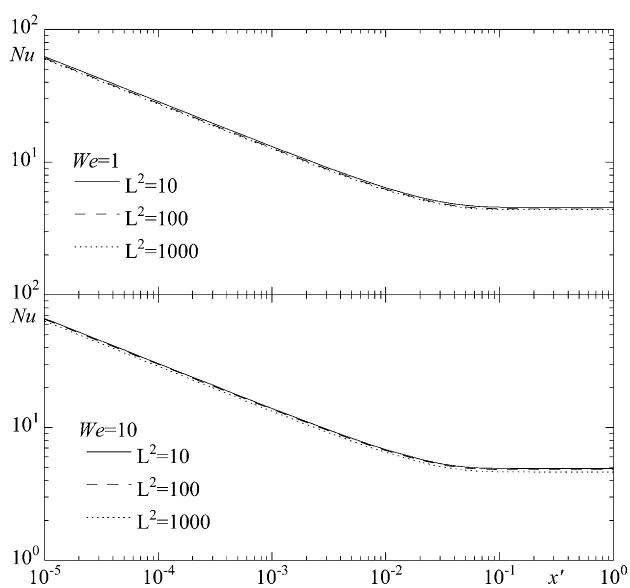

Fig. 1. Effect of viscoelasticity, measured by We and L^2, on the Nusselt number variation in the absence of viscous dissipation ($Br = 0$) in a pipe.

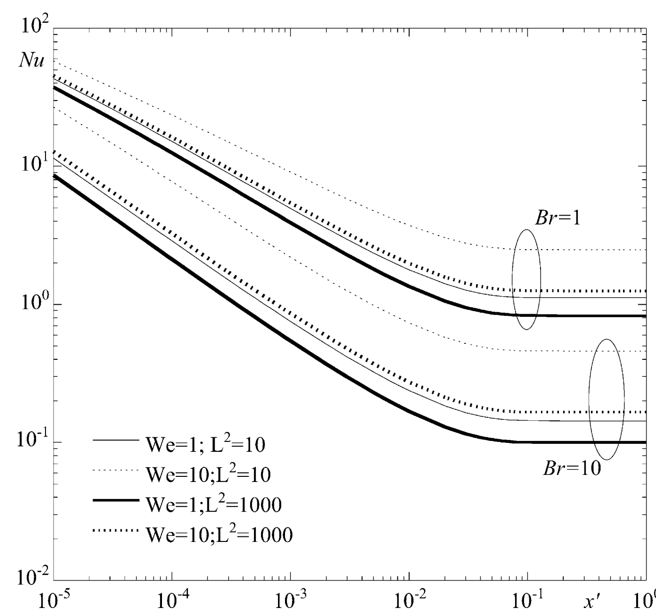

Fig. 2. Effect of viscoelasticity, measured by We and L^2, and Brinkman number on the Nusselt number variation in a pipe (wall heating $q_w > 0$).

In conclusion, we have described a method to obtain thermal entry flow results for the viscoelastic FENE-P model. Nusselt number variations have been given for particular sets of model parameters and one boundary condition (imposed wall heat flux), but the reader may easily use the codes supplied and apply the equivalence between parameters of the PTT and FENE-P models, established by Eq. (10), to obtain results for other conditions. In actual engineering applications the thermal properties cannot be considered independent of the temperature. A sensitivity study of this effect for the PTT fluid has recently been carried out by Nóbrega et al. [9], where guidelines on the required corrections can be found. A final comment on the fact of whether the present results could be obtained from solution of the governing equations for a generalised Newtonian fluid possessing an appropriate viscosity function. While the assertion is correct, it should be realised that the viscosity prescription would be rather complex; in fact it would have to take into account the effect of the normal stress $\boldsymbol{\tau}_{xx}$ variation across the pipe, as in the PTT and FENE-P models, and hence in a sense "elasticity" would be indirectly present.

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