A GNF framework for turbulent flow models of drag reducing fluids and proposal for a k–ε type closure

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Abstract

Based on a generalised Newtonian fluid (GNF) model, modified to account for strain-thickening of the extensional viscosity, this paper derives transport equations for mass, momentum, Reynolds stresses, turbulent kinetic energy and its rate of dissipation. An analysis of order of magnitude identifies the relevant new terms and suggestions are made to model those terms needed to ensure closure in the perspective of a low Reynolds number k–ε model. Specifically, a closed model for the time-average viscosity is proposed that takes into account its non-linearity and dependence on the second and third invariants of the fluctuating rate of deformation tensor. The turbulence model is qualitatively shown to increase the rate of decay of turbulent kinetic energy in isotropic grid turbulence for certain rheological conditions. The performance of the turbulence model in a pipe flow is assessed in a companion paper by Cruz and Pinho [J. Non-Newtonian Fluid Mech., in press].
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Keywords: Single-point closures; Drag reducing fluids; Turbulent flow models; Modified generalised Newtonian fluid

1. Introduction

Since 1948, when Toms [2] first reported the existence of drag reduction in turbulent pipe flow of non-Newtonian fluids, many researchers have dedicated their time at understanding the behaviour of viscoelastic fluids under turbulent flow conditions. Such efforts culminated in the mid 1970s in a fair amount of phenomenological understanding as is well documented in the reviews of Hoyt [3] and Virk [4].

For the next 20 years, research efforts were aimed at a more in-depth physical understanding of the details of such wall-dominated flows, made possible by developments in optical diagnostic techniques. Examples of detailed investigations with non-intrusive optical techniques are the earlier works of Achia and Thompson [5], Reischman and Tiederman [6], those of Luchik and Tiederman [7,8] and Pinho and

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Whitelaw [9] in the 1980s and more recently of Warholic et al. [10], Pereira and Pinho [11] and Escudier et al. [12], amongst others. Note that Warholic et al. [10] investigated heterogenous drag reduction, rather than the homogeneous form of concern here and in the other works listed, but some of their findings are equally relevant. In 1995 Gyr and Bewersdorff [13] reviewed critically the existent knowledge not only for polymer solutions but also for surfactants and fibre suspensions.

A major finding of Tiederman's and Hanratty's experimental investigations on channel flows was the dramatic reduction of the Reynolds shear stress. The ensuing shear stress deficit must be accounted for by the appearance of an extra elastic shear stress which has not yet been directly measured. This finding suggests a new turbulence dynamics involving, amongst others, the coupling between fluctuations of elastic stresses and fluctuating velocity gradients. This was confirmed by the DNS investigation of Massah and Hanratty [14] with the FENE-P model who found that the added polymer stresses fluctuate and interact with the turbulence and the mean flow. Under certain conditions the effect is similar to an increased dissipation bearing some resemblance to the effect of anisotropic viscosity on stresses, and contributing to drag reduction. In fact, when decoupling the shear stress into viscous, inertial and polymer components, the latter can assume both positive and negative values, i.e. it can act as a source as well as a sink of turbulent kinetic energy (see also the recent experimental results of Ptasinski et al. [15]).

In spite of all the efforts there is still no clear explanation for the relationship between the observed flow characteristics and the rheology of the viscoelastic fluids. However, there are strong indications for a relationship between drag reduction and extensional viscosity, but so far there has been no definite proof of this speculation, one reason being the difficulties in measuring the extensional viscosity of dilute and semi-dilute solutions required to attain turbulent flows.

The availability of increased computer power has offered an important research alternative by providing the means for direct numerical simulations (DNS). Starting with Massah et al. [16]. DNS have provided useful information on the effect of specific fluid properties on turbulent flow characteristics and is presently the most powerful method available for probing the physics of turbulent flow. For viscoelastic fluids there is, however, one important difficulty relative to DNS with Newtonian fluids: a priori there is no certainty to what is the correct rheological constitutive equation for a given fluid. Even so, DNS is providing useful insight that will enable researchers to select adequate constitutive equations and, more important, to develop useful and more accurate single-point closures for classical or newly developed turbulence models.

The initial DNS investigations were not self-consistent as they only solved the constitutive equation, usually the FENE-P model, for fixed Newtonian kinematics. This strategy, adopted by Hanratty and co-workers [14,16], is not able to predict drag reduction but gave insight onto the evolution of the molecular configurations and the corresponding fluid properties, with the turbulence dynamics. One of their main findings were the large molecular extensions in the viscous sublayer but no significant molecular extensions in the buffer layer. Since in a laminar Couette flow, the molecules were also significantly extended but no drag reduction was found, it was concluded that although molecular extension affected the shear and elongational viscosities, drag was only a function of the shear stress which is not affected by the extensional viscosity in laminar flow. However, extensional viscosity interferes with the dynamics of turbulence and consequently the shear stress in turbulent flow is affected.

Progress in computer technology enabled the first self-consistent DNS, but the complexity of viscoelastic differential constitutive equations led researchers to adopt simpler rheological equations aimed at assessing the effects of particular rheological properties. This was done by Orlandi [17] and Den Toonder et al. [18,19], who adopted viscous constitutive equations to mimic the effects of polymers in DNS. In Orlandi the anisotropy of a polymer viscous contribution to normal stresses was related to the magnitudes

of strain and rotation rate tensors to mimic the effect of molecular extension and this was found to be sufficient to create drag reduction. Den Toonder et al. [18] investigated two constitutive models: in model 1, an isotropic normal stress polymer contribution was only sensitive to stretching of the molecules and, in model 2, the extensional viscosity increased with both stretching and compression of molecules. Only the latter model lead to drag reduction. In Den Toonder et al. [19] an anisotropic viscous model and an anisotropic simple viscoelastic model, both made especially sensitive to elongational effects, were investigated. In this work the anisotropic viscous model was able to predict higher drag reductions than the viscoelastic model.

More recently, self-consistent DNS investigations of turbulent channel flow with viscoelastic constitutive equations derived from kinetic (FENE-P) and network (Giesekus) theories have been carried out by Sureshkumar et al. [20] and Dimitropoulos et al. [21,22]. These works were able to predict drag reduction and have shown qualitative agreement with experimental findings. In fact, the simulations with the FENE-P and Giesekus models showed similar amounts of drag reduction when their parameters were chosen to match the plateau extensional viscosity [22]. Dimitropoulos et al. [22] have also provided budgets of turbulence kinetic energy, Reynolds stresses and vorticity and consequently it is a major reference for the development of single-point turbulence closures. Drag reduction was found to be directly related to the extensibility of the polymer chains and that a pre-requisite for drag reduction is a sufficiently enhanced elongational viscosity in agreement with the findings from various other sources. The recent experiments of Ptasinski et al. [15] also provided budgets of mean energy and of turbulent kinetic energy, the results of which confirm some findings by DNS.

As in Newtonian turbulent flow, DNS is a powerful research tool but requires enormous computing resources and is not practical for calculations of industrial flows. For this purpose, single-point turbulence closures are required. In contrast to experimental work, progress in single-point turbulence models for viscoelastic flow predictions has been rather slow. After an initial effort by various research groups in the late 1970s [23–26] very few developments have taken place henceforth. In some of these initial investigations, the k–ε turbulence model for Newtonian fluids was used with very specific modifications in wall functions (standard model) or damping functions (low Reynolds number model). In Hassid and Poreh [24] a one-equation model was suggested, but in general these were rapidly being discarded for better models due to difficulties in defining the appropriate length scale. In Poreh and co-workers [25,26] the same modified version of the low Reynolds number k–ε model of Jones and Launder [27] was used. However, this was not sufficient for predicting drag reduction and the coefficient A of the Van Driest type of damping function for eddy viscosity had to be determined from the results of drag reduction at a particular Reynolds number for a specific fluid and pipe geometry to enable the accurate calculation of the flow characteristics. A similar strategy was adopted by Durst and Rastogi [23].

In essence, these modifications were unable to deal with drag reduction with generality, but showed that adequate modifications of the law of the wall or of damping functions would lead to correct predictions. This deficiency stems from the lack of connection between turbulence model and fluid rheology. A solution to this shortcoming was attempted as early as 1973 by Mizushina et al. [28]: the Van Driest damping factor in a zero equation turbulence model, was modified to account for the viscoelasticity of the fluid by incorporating Rouse's relaxation time [29] to be determined from experiments. This approach enabled predictions of several flow conditions with the same fluid, a situation that not even the later models of Hassid and co-workers [24–26] and Durst and Rastogi [23] could achieve.

More recent excursions into the same subject were also of limited application: Politis [30], Cruz et al. [31] and Malin [32,33] independently deduced the relevant k–ε equations for turbulent flows of inelastic

power law fluids but the intense drag reduction of elastic origin could not be predicted. However, some of their ideas are pursued in the present work and its continuation [1].

Developments of eddy viscosity models have also been attempted. Poreh and Dimant [34] developed a model based on Van Driest's mixing length with a variable damping parameter to represent the effect of drag reducing additives, and more complex expressions were derived by Edwards and Smith [35] and recently by Azouz and Shirazi [36]. In the latter, the purpose was to predict turbulent flow of polymer solutions in annuli but again the model required previous measurements in pipe flow with the same fluids.

There is clearly the need for further progress in turbulence modelling for viscoelastic fluids, in particular taking into account the extensional behaviour of the drag reducing fluids. This paper constitutes the first step of a work aimed at developing and testing turbulence models for drag reducing fluids obeying a specific constitutive equation. Specifically, a generalised Newtonian fluid model, that was modified to mimic some effects of the extensional viscosity, was chosen as constitutive equation.

Instead of adopting immediately a complex differential constitutive equation, for which modelling would be more difficult, a generalised Newtonian fluid modified to mimic extensional viscosity enhancement effects will lead to a turbulence model with many similarities to a Newtonian turbulence model. Turbulence models for Newtonian fluids do not suffer from the difficulty of choosing a constitutive equation and yet they often rely on damping functions and other approaches to compensate for inadequate or incomplete physics modelling. There is no reason to assume that it will not be so for viscoelastic fluids. If a more complex, but faithful, viscoelastic constitutive equation is adopted, it is more likely that there will be more ad hoc assumptions and simplifications than if a simpler Newtonian-like rheological expression is used. This will provide us with the experience for more elaborate turbulent closures in the future.

This first paper is rather general and is aimed at deriving the transport equations needed in single-point first- and second-order turbulence models. First, the constitutive model adopted and its modifications are explained and the corresponding time-averaged conservation equations of mass, momentum, turbulent kinetic energy and its rate of dissipation are presented (their derivation is presented in Appendix A). An analysis of order of magnitude is then carried out on those equations to identify relevant new terms and proposals are made for closure especially regarding a low Reynolds number k–ε model. At the end the new k–ε model is tested qualitatively in isotropic decay of grid generated turbulence. In a second companion paper [1] the k–ε model is further developed for wall flows and its capacities are investigated via predictions of pipe flow and comparisons with experimental data from the literature.

2. A constitutive equation

The most important rheological property of polymer solutions that must be taken into account is the viscometric viscosity. It can be constant, as with Newtonian fluids, but most often exhibits some degree of shear-thinning. A generalised Newtonian fluid model (GNF) is adequate to predict this fluid property accurately.

For a long time there was controversy as to what rheological properties caused drag reduction and this was discussed in Oliveira and Pinho [37]. There, the relevance of extensional viscosity was made clear and the recent DNS works of Dimitropoulos et al. [21,22] and De Angelis et al. [38] have extensively confirmed it.

Experimentally, Escudier et al. [12] were probably the first to measure drag reduction and detailed velocity profiles with polymer solutions for which they provided elongational viscosity data measured

with an opposed-jet nozzle rheometer. Their measurements with different types of polymers showed shear-thinning of the viscometric viscosity and confirmed the strain-thickening of the Trouton ratio, relevant rheological features for drag reduction [37].

As mentioned in the introduction, the constitutive equation adopted is simple: the GNF model modified to mimic extensional viscosity strain-thickening. This was the model used by Oliveira and Pinho [37], here with small modifications. The elongational viscosity is introduced into the GNF constitutive equation by making it a function of the strain rate $\dot{\varepsilon}$ as explained in more detail by Oliveira and Pinho [37]. The GNF fluid, with dependence on the shear rate $\dot{\gamma}$ and strain rate $\dot{\varepsilon}$, is written as

$$\sigma = 2\mu \mathbf{S}, \tag{1}$$

where μ is the viscosity function and $\mathbf{S}$ is the rate of deformation tensor defined by

$$\mathbf{S} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2}$$

The viscosity function

$$\mu = \mu(\dot{\gamma}, \dot{\varepsilon}), \tag{3}$$

depends on $\dot{\gamma}$ and $\dot{\varepsilon}$ which are related to the invariants of $\mathbf{S}$ in the following way:

$$\dot{\gamma} = \sqrt{-4II_{\mathbf{S}}} = \sqrt{2S_{ij}S_{ij}}, \tag{4}$$

$$\dot{\varepsilon} = \frac{6\det\mathbf{S}}{\operatorname{tr}\mathbf{S}^2} = \frac{2\operatorname{tr}\mathbf{S}^3}{\operatorname{tr}\mathbf{S}^2} = \frac{2(S_{ik}S_{kj})S_{ij}}{S_{ij}S_{ij}}. \tag{5}$$

An algebraic form for the viscosity function (3) can be a Bird–Carreau type of equation

$$\mu = \mu_0[1 + (\lambda_{\mathrm{s}}\dot{\gamma})^2]^{(n-1)/2}[1 + (\lambda_{\mathrm{e}}\dot{\varepsilon})^2]^{(p-1)/2}, \tag{6}$$

but, for simplicity in the derivation of the turbulence model, a power law-based equation is preferred. Thus, the viscosity equation adopted henceforth is

$$\mu = K_{\mathrm{v}}[\dot{\gamma}^2]^{(n-1)/2}K_{\mathrm{e}}[\dot{\varepsilon}^2]^{(p-1)/2}, \tag{7}$$

where some constraints to the various parameters are imposed later. The model is the product of a shear rate dependent term with a strain rate dependent term. Both terms do not have to be dimensionally identical but their product must be a viscosity and Eq. (7) must also obey some limiting conditions and agree with rheological measurements. The specific meanings of the various model parameters are specified in Cruz and Pinho [1], and at this stage it is only important to consider K_{v}, K_{e}, n and p as known fluid properties. In any case, the model pretends to represent a behaviour where there is shear-thinning due to dependence on $\dot{\gamma}$ ($n < 1$, $p = 1$) and strain-thickening due to the dependence on $\dot{\varepsilon}$ ($n = 1$, $p > 1$) as is sketched in Fig. 1.

For this GNF constitutive equation it is now necessary to deduce the corresponding conservation equations for turbulent flow, bearing in mind that there are fluctuations in the viscosity because of its non-linear dependence on the flow kinematics. Although specific viscosity expressions are presented in Eqs. (6) and (7), the work in Section 3, and in Appendix A, is totally independent of the adopted equation for μ. A specific viscosity model is only required from Section 4 onwards, but even then part of this work remains general and independent of specific forms adopted for μ.
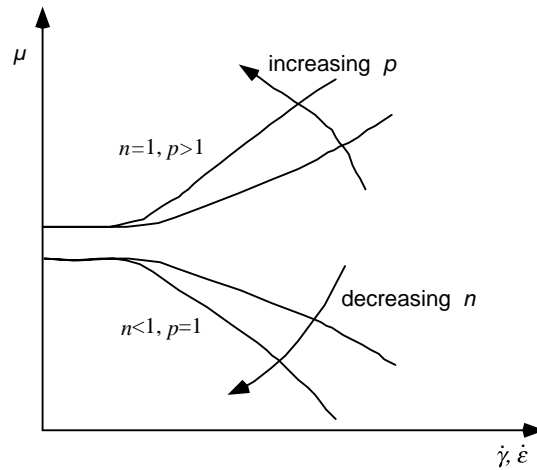
Fig. 1. Schematic representation of the effect of *n* and *p* on the variation of the viscosity with the shear and strain rates in log–log coordinates.

3. Conservation equations

Modern developments of one-point closures for turbulence are usually based on first- or second-order models. Since an objective of this work is to establish a framework for developing a specific type of couple turbulence rheology closure, conservation equations for mass, momentum, turbulent kinetic energy, its rate of dissipation and for the Reynolds stress tensor need to be derived for these fluids of variable viscosity, where the first major conceptual difference relative to a Newtonian fluid is the existence of fluctuations of viscosity.

The derivation of all transport equations is tedious and has similarities to that for Newtonian fluids. In order not to break the logical sequence of the text, such work is left to Appendix A and here only the final forms of the various equations are presented. Throughout the paper the Reynolds decomposition is used and the average of the fluctuating quantities, including the viscosity, is zero. Capital letters or an overbar designate average values, small letters or a prime designate fluctuating quantities, the exception being the average pressure represented by *p*.

The Reynolds-averaged momentum equation for a GNF fluid is

$$\rho \frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial (2\bar{\mu} S_{ik} + 2\overline{\mu' s_{ik}} - \rho \overline{u_i u_k})}{\partial x_k}. \tag{8}$$

Relative to the momentum equation for a Newtonian fluid there is a new diffusive term $(2\overline{\mu' s_{ik}})$ and the classical term $(2\bar{\mu} S_{ik})$ is modified. Both need to be evaluated later for closure. As will be seen in Section 4, $\bar{\mu}$ also depends on s_{ij} and S_{ij} although at high Reynolds numbers the dependence on s_{ij} is more important.

In recent experimental and DNS investigations of turbulent duct flows of viscoelastic fluids using the FENE-P model ([14,15,21,22] amongst others), the total stress is written as the sum of the solvent, the polymer and the Reynolds shear stress tensors: $\tau_{ij} = \tau_{ij,s} + \tau_{ij,p} - \rho \overline{u_i u_j}$. It is important to understand that $\tau_{ij,p}$ accounts not only for an elastic contribution but also for a possibly large viscous contribution.

In our formulation, however, the separation of effects is different because both $2\bar{\mu}S_{ij}$ and $2\overline{\mu's_{ij}}$ include purely viscous and extensional contributions of the polymer solution (notice that $\overline{\mu's_{ij}} \neq 0$ for an inelastic shear-thinning fluid). Thus, $\tau_{ij,\mathrm{s}}$ is totally included in $2\bar{\mu}S_{ij}$ but so is also part of $\tau_{ij,\mathrm{p}}$.

The Reynolds stress transport equation is given by

$$
\rho\frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} + \rho\overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \rho\overline{u_i u_k}\frac{\partial U_j}{\partial x_k}
$$
$$
= -\rho\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} - \left(\frac{\partial}{\partial x_i}\overline{p'u_j} + \frac{\partial}{\partial x_j}\overline{p'u_i}\right) + \overline{p'\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)} + \bar{\mu}\frac{\partial^2\overline{u_i u_j}}{\partial x_k \partial x_k}
$$
$$
- 2\bar{\mu}\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \frac{\partial\bar{\mu}}{\partial x_k}\frac{\partial\overline{u_i u_j}}{\partial x_k} + \frac{\partial\bar{\mu}}{\partial x_k}\left(\frac{\partial\overline{u_k u_j}}{\partial x_i} + \frac{\partial\overline{u_k u_i}}{\partial x_j} - 2\overline{u_k s_{ij}}\right) + \mu'\overline{\frac{\partial^2 u_i u_j}{\partial x_k \partial x_k}}
$$
$$
- 2\overline{\mu'\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\frac{\partial u_i u_j}{\partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\left(u_j\frac{\partial u_k}{\partial x_i} + u_i\frac{\partial u_k}{\partial x_j}\right)} + \overline{\mu'u_j}\frac{\partial^2 U_i}{\partial x_k \partial x_k}
$$
$$
+ \overline{\mu'u_i}\frac{\partial^2 U_j}{\partial x_k \partial x_k} + \overline{u_j\frac{\partial\mu'}{\partial x_k}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)} + \overline{u_i\frac{\partial\mu'}{\partial x_k}\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right)}, \tag{9}
$$

and contraction of indices gives the transport equation for the turbulent kinetic energy ($k \equiv \overline{u_{ii}^2}/2$):

$$
\rho\frac{\mathrm{D}k}{\mathrm{D}t} = -\frac{\partial\overline{u_j p}}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho\overline{u_i u_j u_i} - 2\bar{\mu}\overline{u_i s_{ij}} - 2\overline{\mu'u_i}S_{ij} - 2\overline{\mu'u_i s_{ij}}\right]
$$
$$
- 2\bar{\mu}\overline{s_{ij}^2} - 2\overline{\mu's_{ij}^2} - 2\overline{\mu's_{ij}}S_{ij} - \rho\overline{u_i u_j}S_{ij}. \tag{10}
$$

Eqs. (9) and (10) include various non-Newtonian terms, but even some of the Newtonian terms are presented in an unusual way. The reader can recover the classical equation found in many papers and textbooks bearing in mind properties due to the symmetry of the rate of deformation tensor S_{ij} (see also Appendix A).

3.1. The rate of dissipation of turbulent kinetic energy

For Newtonian fluids the average rate of dissipation (ε) of turbulent kinetic energy (k), per unit of mass, is defined as

$$
\rho\varepsilon_n \equiv 2\mu\overline{s_{ij}^2}, \tag{11}
$$

where the subscript n is used here to distinguish it from the rate of dissipation of k for GNF fluids. Eq. (11) is the time-average of the instantaneous rate of dissipation defined by

$$
\rho\hat{\varepsilon}_n \equiv 2\mu s_{ij}^2. \tag{12}
$$

By analogy, one can define an instantaneous rate of dissipation for non-Newtonian fluids obeying the GNF model by using the instantaneous viscosity:

$$
\rho\hat{\varepsilon} \equiv 2\hat{\mu}s_{ij}^2. \tag{13}
$$

Time-averaging equation (13) provides the average rate of dissipation for the GNF fluid

$$\rho\varepsilon = 2\overline{\hat{\mu}s_{ij}^2} = 2\overline{(\bar{\mu} + \mu')s_{ij}^2} = 2\bar{\mu}\overline{s_{ij}^2} + 2\overline{\mu's_{ij}^2}. \tag{14}$$

These definitions of instantaneous and average rates of dissipation are identical to those used by Politis [30] in his derivation of a k–ε model of turbulence for purely viscous shear-thinning fluids.

The definition of the rate of dissipation of turbulent kinetic energy also deserves a comment in light of the literature involving the FENE-P or similar models. Several works reviewed in Section 1 reported the existence of a deficit in Reynolds shear stress and the consequent existence of a polymer stress, here called τ_{ij}^P. The fluctuations of τ_{ij}^P contribute to increase or decrease turbulence via the term $-\overline{\tau_{ij}'^P s_{ij}}$ which can take positive or negative values, respectively [10]. Also, in a viscoelastic formulation the transport equations of k and $\overline{u_i u_j}$ contain a term for the interaction between the elastic stress and the rate of strain as shown in the DNS investigation for a FENE-P fluid of Dimitropoulos et al. [22] (their ε_{ij}^v term). Their results do confirm that ε_{ij}^v (in combination to the interaction between $\tau_{ij}'^P$ and u_i) acts as a turbulence production term near the wall and as dissipation elsewhere.

By defining the total stress tensor as $\tau_{ij} = 2\bar{\mu}S_{ij} - 2\overline{\mu's_{ij}} - \rho\overline{u_i u_j}$ and the rate of dissipation as in Eq. (14), its two terms in Eq. (10) already include such interaction between fluctuating stress and fluctuating shear rate. ε is defined as a single quantity but still it is the sum of a positive definite term ($2\bar{\mu}\overline{s_{ij}^2}$) with a term that can be either a source or a sink of dissipation ($2\overline{\mu's_{ij}^2}$). So, in a limiting situation of a 1D shear flow, where the Reynolds shear stresses are found to be negligible (due to the presence of additives, cf. [10]), under the classical equilibrium condition the definition of Eq. (14) results in negligible production of turbulence and negligible dissipation. However, for the flow to be turbulent there must be production of turbulence, as well as dissipation, and so what is really happening is $-2\overline{\mu's_{ij}^2} = 2\bar{\mu}\overline{s_{ij}^2}$. Note that recently Ptasinski et al. [15] showed that in pipe flow the Reynolds stress stays definitely non-zero even at maximum drag reduction.

3.2. The transport equation for the rate of dissipation of turbulent kinetic energy

$$\underbrace{\rho\frac{\partial\varepsilon}{\partial t}}_{\text{Ia}} + \underbrace{\rho U_k\frac{\partial\varepsilon}{\partial x_k}}_{\text{Ib}} = \underbrace{-2\frac{\partial U_i}{\partial x_k}\left[\overline{(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}}\right]}_{\text{IIa}} - \underbrace{2\frac{\partial U_k}{\partial x_m}\left[\overline{(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}}\right]}_{\text{IIb}}$$

$$\underbrace{-2\frac{\partial^2 U_i}{\partial x_k \partial x_m}\left[\overline{(\bar{\mu}+\mu')u_k\frac{\partial u_i}{\partial x_m}}\right]}_{\text{IIc}} - \underbrace{2\overline{(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}}}_{\text{IId}}$$

$$\underbrace{-(\bar{\mu}+\mu')\frac{\partial}{\partial x_k}\left[\overline{u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right]}_{\text{IIIa}} - \underbrace{2(\bar{v}+v')\frac{\partial}{\partial x_i}\left[\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right]}_{\text{IIIb}}$$

$$\underbrace{+\frac{(\bar{\mu}+\mu')^2}{\rho}\frac{\partial}{\partial x_k}\left[\overline{\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)}\right]}_{\text{IV}} - \underbrace{2\frac{(\bar{\mu}+\mu')^2}{\rho}\overline{\frac{\partial^2 u_i}{\partial x_k \partial x_m}\frac{\partial^2 u_i}{\partial x_k \partial x_m}}}_{\text{V}}$$

$$+ \underbrace{4(\bar{\nu} + \nu')\frac{\partial^2(\bar{\mu} + \mu')}{\partial x_m \partial x_k}\left[\frac{\partial u_i}{\partial x_m}s_{ik}\right]}_{\text{VI}} + \underbrace{\overline{4(\bar{\nu} + \nu')\frac{\partial(\bar{\mu} + \mu')}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial s_{ik}}{\partial x_m}}}_{\text{VII}}$$

$$+ \underbrace{2(\bar{\nu} + \nu')\frac{\partial(\bar{\mu} + \mu')}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}_{\text{VIII}} + \underbrace{4S_{ik}\left[\overline{(\bar{\nu} + \nu')\frac{\partial u_i}{\partial x_m}\frac{\partial^2 \mu'}{\partial x_m \partial x_k}}\right]}_{\text{IX}}$$

$$+ \underbrace{4\frac{\partial S_{ik}}{\partial x_m}\left[\overline{(\bar{\nu} + \nu')\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_k}}\right]}_{\text{X}} + \underbrace{2\frac{\partial^2 U_i}{\partial x_k \partial x_k}\left[\overline{(\bar{\nu} + \nu')\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_m}}\right]}_{\text{XI}}$$

$$+ \underbrace{2\frac{\partial}{\partial x_m}\left(\frac{\partial^2 U_i}{\partial x_k \partial x_k}\right)\left[\overline{(\bar{\nu} + \nu')\mu'\frac{\partial u_i}{\partial x_m}}\right]}_{\text{XII}} - \underbrace{4\nu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \mu'}{\partial x_k}s_{ik}\right]}_{\text{XIIIa}}$$

$$- \underbrace{\overline{2\nu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\mu'\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right]}}_{\text{XIIIb}} + \underbrace{\overline{2\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \overline{u_i u_k}}{\partial x_k}\right]}}_{\text{XIV}}$$

$$+ \underbrace{\overline{\frac{\partial(\bar{\mu} + \mu')}{\partial t}\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}}_{\text{XVa}} + \underbrace{U_k\frac{\overline{\partial(\bar{\mu} + \mu')}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}}_{\text{XVb}}. \tag{15}$$

The transport equation for the time-average rate of dissipation (ε) of turbulent kinetic energy of a GNF fluid is given by Eq. (15).

In Eq. (15) the alternative definitions of the average and instantaneous rates of dissipation are $\rho\varepsilon \equiv \overline{(\bar{\mu} + \mu')(\partial u_i/\partial x_m)(\partial u_i/\partial x_m)}$ and $\rho\hat{\varepsilon} \equiv (\bar{\mu} + \mu')(\partial u_i/\partial x_m)(\partial u_i/\partial x_m)$, respectively. These two definitions of rate of dissipation are equivalent to those of Eqs. (13) and (14) only under the assumption of homogeneous turbulence. For convenience, in writing down Eq. (15) use was made of the kinematic viscosity ($\nu \equiv \mu/\rho$).

Eq. (15) includes two types of terms: those having similarities with terms found in the dissipation equation for a Newtonian fluid originate from the inertial, pressure and one of the viscous terms in the momentum equation, and include both the average and fluctuating viscosities. The second set of terms is new and involves the fluctuating viscosity μ'. Often, their physical meaning can be easily identified due to similarities with terms involving the average viscosity.

4. The time-average molecular viscosity

In turbulent flow, the molecular viscosity of a variable-viscosity fluid depends on the fluctuating rates of shear and strain, a major difference relative to a Newtonian fluid. Hence, a major contribution of this work is the relationship between the time-average molecular viscosity and turbulent quantities to ensure closure of the set of equations. In order to arrive at such relationship it is now necessary to adopt a specific form for the viscosity function and here Eq. (7) is used.

In terms of instantaneous values the viscosity is

$$\hat{\mu} = K_{\mathrm{v}}[\hat{\dot{\gamma}}^2]^{(n-1)/2} K_{\mathrm{e}}[\hat{\varepsilon}^2]^{(p-1)/2}, \tag{16}$$

with $\hat{\dot{\gamma}}$ and $\hat{\varepsilon}$ following from Eqs. (4) and (5).

The maximum value of $\hat{\varepsilon}$ was estimated by Oliveira and Pinho [37] to be

$$\hat{\varepsilon}_{\max} = \sqrt{\tfrac{2}{3} \hat{S}_{ij} \hat{S}_{ij}}. \tag{17}$$

For a high Reynolds number turbulent flow Tennekes and Lumley [39] have shown that $\hat{S}_{ij} \hat{S}_{ij} \cong s_{ij} s_{ij}$, thus Oliveira and Pinho [37] concluded that

$$\hat{\varepsilon}_{\max} = \sqrt{\tfrac{2}{3} s_{ij} s_{ij}}. \tag{18}$$

Typical values of $\hat{\varepsilon}$ being smaller, let us assume that in general

$$\hat{\varepsilon} = \frac{\sqrt{s_{ij} s_{ij}}}{A_{\varepsilon}}, \tag{19}$$

where the value of A_{ε} is to be found from experimental data (see [1]) but must be higher than $\sqrt{3/2}$.

Back-substituting these definitions into the viscosity model and simplifying gives

$$\hat{\mu} = \frac{K_{\mathrm{v}} K_{\mathrm{e}}}{A_{\varepsilon}^{p-1}} 2^{(n-1)/2} [s_{ij}^2]^{(n+p-2)/2}. \tag{20}$$

This expression can now be used to calculate the instantaneous rate of dissipation in Eq. (13):

$$\rho \hat{\varepsilon} = 2^{(n+1)/2} \frac{K_{\mathrm{v}} K_{\mathrm{e}}}{A_{\varepsilon}^{p-1}} [s_{ij}^2]^{(n+p)/2}. \tag{21}$$

Eqs. (20) and (21) are combined to eliminate s_{ij}^2 and yielding a relation between the instantaneous viscosity and rate of dissipation

$$\hat{\mu} = \left[\frac{K_{\mathrm{v}} K_{\mathrm{e}}}{A_{\varepsilon}^{p-1}} \right]^{2/(n+p)} 2^{(1-p)/(n+p)} (\rho \hat{\varepsilon})^{(n+p-2)/(n+p)}. \tag{22}$$

Introducing parameters

$$m \equiv \frac{n+p-2}{n+p}, \tag{23a}$$

and

$$B \equiv \left[\frac{K_{\mathrm{v}} K_{\mathrm{e}}}{A_{\varepsilon}^{p-1}} \right]^{1-m} 2^{((n-1)-m(n+1))/2} \rho^m, \tag{23b}$$

for compactness, Eq. (22) assumes the simple form

$$\hat{\mu} = B \hat{\varepsilon}^m. \tag{24}$$

The average viscosity and the average rate of dissipation are determined using their probability distribution functions. By definition, the time-average viscosity

$$\bar{\mu} = \int_0^\infty B\hat{\varepsilon}^m P(\hat{\varepsilon}) \, \mathrm{d}\hat{\varepsilon}. \tag{25}$$

Since the instantaneous viscosity is always a positive quantity, the instantaneous rate of dissipation ($\hat{\varepsilon}$) is positive-definite and is associated with small scale motion, assumed here to be locally isotropic at high Reynolds number flows. Thus, as explained by Monin and Yaglom [40], $\hat{\varepsilon}$ follows a log-normal distribution

$$P(\hat{\varepsilon}) = \frac{1}{\hat{\varepsilon}\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \hat{\varepsilon} - M}{\sigma} \right)^2 \right\}, \tag{26}$$

with M and σ standing for the mean and standard deviation of $\hat{\varepsilon}$.

Following Monin and Yaglom [40] (pp. 614–615),

$$\int_0^\infty \frac{\hat{\varepsilon}^m}{\hat{\varepsilon}\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \hat{\varepsilon} - M}{\sigma} \right)^2 \right\} \, \mathrm{d}\hat{\varepsilon} = \exp\left(mM + \frac{m^2\sigma^2}{2} \right), \tag{27}$$

so, the average viscosity comes out as

$$\bar{\mu} = B \exp(mM + \tfrac{1}{2}m^2\sigma^2). \tag{28}$$

The average rate of dissipation is also obtained from the probability distribution function of $\hat{\varepsilon}$

$$\varepsilon = \int_0^\infty \hat{\varepsilon} P(\hat{\varepsilon}) \, \mathrm{d}\hat{\varepsilon} = \int_0^\infty \hat{\varepsilon} \frac{1}{\hat{\varepsilon}\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \hat{\varepsilon} - M}{\sigma} \right)^2 \right\} \, \mathrm{d}\hat{\varepsilon}_k = \exp\left(M + \frac{\sigma^2}{2} \right). \tag{29}$$

Now, $\bar{\mu}$ and ε can be related to each other by solving Eq. (28) to get e^M and substituting it back into Eq. (29). The result is

$$\bar{\mu} = B\varepsilon^m \, \mathrm{e}^{m\sigma^2(m-1)/2}, \tag{30}$$

with m and B given by Eqs. (23a) and (23b).

In Eq. (30), σ^2 is the variance of the distribution of $\ln \hat{\varepsilon}$ which is given by

$$\sigma^2 = A_1 + A_2 \ln\left(\frac{L}{\eta} \right), \tag{31}$$

with L representing an external turbulence length scale, such as a large eddy scale, and η an appropriately defined internal length scale such as the Kolmogorov microscale [40]. A_1 is a parameter depending on the turbulence macrostructure and A_2 is a universal constant. The extensive discussion of these quantities, in pp. 612–626 of Monin and Yaglom [40], suggests that $A_2 = 0.3$–0.5, although the most appropriate value seems to be between 0.4 and 0.5, thus 0.45 is considered. A_1 depends on the form of the space regions and there is very little information concerning it (p. 634 in [40]), so it is assumed to be zero. Hence, σ^2 is finally given by

$$\sigma^2 = A_2 \ln\left(\frac{L}{\eta} \right). \tag{32}$$

The Kolmogorov scale is

$$\eta \equiv \left[\frac{\bar{v}^3}{\varepsilon}\right]^{1/4} = \left[\frac{\bar{\mu}^3}{\rho^3 \varepsilon}\right]^{1/4}, \tag{33}$$

where the average molecular viscosity was used. The large length scale for the energy containing eddies is given by the inviscid estimate proportional to $k^{1.5}/\varepsilon$. A commonly used equation for L derived from inviscid theory arguments is adopted here as given by Younis [41]

$$L \approx \frac{2C_\mu^{0.75} k^{1.5}}{\varepsilon}, \tag{34}$$

where C_μ is a universal constant that usually assumes the numerical value of 0.09 for Newtonian fluids.

Finally, combining Eqs. (30)–(34) provides the final form of an explicit expression for $\bar{\mu}$

$$\bar{\mu} = (C_\mu \rho)^{3m(m-1)A_2/(8+3m(m-1)A_2)} 2^{4m(m-1)A_2/(8+3m(m-1)A_2)} k^{6m(m-1)A_2/(8+3m(m-1)A_2)}$$
$$\varepsilon^{[8-3(m-1)A_2]m/(8+3m(m-1)A_2)} B^{8/(8+3m(m-1)A_2)} \tag{35}$$

with m and B defined above. In the limiting case of a Newtonian fluid ($n = p = l$) a constant average molecular viscosity of $\bar{\mu} = K_v K_e$ is recovered.

With this relationship the full closure of the set of transport equations can be ensured provided the new non-Newtonian terms of the conservation equations are adequately modelled. This is carried out in the next sections, but only in the context of a first-order k–ε turbulence closure. In the companion paper [1] the final details of the model are derived and the model is used to make predictions and comparisons with data from the literature. The developments for more elaborate first-order and second-order models are left for the future.

5. Order of magnitude analysis

An order of magnitude analysis is carried out for all the transport equations to help in their simplification. There will be a preoccupation to identify similarities with the corresponding equations for Newtonian fluids and, because of the lack of experience and knowledge on turbulence modelling for viscoelastic fluids, the number of modifications is kept to a minimum except when based on solid arguments.

To perform the order of magnitude analysis, the following scales are used: L represents a large length scale of the energy containing eddies, U is the velocity scale of mean flow, u is the velocity scale of fluctuations ($u \approx \sqrt{k}$) and l is the length scale associated with small fluctuations and its gradients, which is related to the Kolmogorov scale. Note also that the inviscid estimate of the rate of dissipation is used ($\varepsilon = u^3/L$) and that both the instantaneous molecular viscosity $\hat{\mu}$ and the molecular viscosity fluctuations μ' are needed. This analysis starts with an estimate of these two viscous quantities, but henceforth their kinematic equivalents $\hat{v}$ and v' ($v \equiv \mu/\rho$) are used instead.

The ratio of instantaneous to average molecular viscosities is needed and is estimated as

$$\frac{\hat{v}}{\bar{v}} = \frac{\hat{\varepsilon}^m}{\varepsilon^m \exp[m\sigma^2(m-1)/2]} \sim \exp\left[\frac{m\sigma^2(1-m)}{2}\right], \tag{36}$$

because ε is determined from the pdf of $\hat{\varepsilon}$ and m (Eq. (23)) is a coefficient of order 1 (it depends on n and p where usually $n \leq 1$ and $p \geq 1$). Upon substitution of σ^2 (Eq. (32))

$$\frac{\hat{v}}{\bar{v}} \sim \left(\frac{L}{\eta}\right)^{0.225m(1-m)}. \tag{37}$$

For convenience $a \equiv 0.225m(1 - m)$ is defined. Considering typical values of n and p, $m = 0$ for Newtonian fluids and for non-Newtonian fluids for which the shear-thinning exponent n differs from 1 by exactly the same amount that the strain-thickening p differs from 1 (i.e. when $p = 1 - n$). For other realistic values of n and p [1] the exponent a stays in the range limited by -0.05 and $+0.05$. Finally, considering the Kolmogorov scale (Eq. (33)) and L from Eq. (34):

$$\frac{\hat{v}}{\bar{v}} \sim 0.33^a \left(\frac{uL}{\bar{v}}\right)^{3a/4} \sim \left(\frac{uL}{\bar{v}}\right)^{3a/4}, \tag{38}$$

and, by definition,

$$\frac{v'}{\bar{v}} \sim \left(\frac{uL}{\bar{v}}\right)^{3a/4} - 1. \tag{39}$$

The use of the difference in estimating v' is advantageous, because it will produce a null contribution for fluids having a constant viscosity.

5.1. Momentum equation

The momentum equation (Eq. (8)) has a modified diffusive term that can now be calculated since there is a relationship for $\bar{\mu}$ and a new diffusive term ($2\overline{\mu' s_{ik}}$) that must be evaluated.

The relevance of this new term is assessed by comparison with the modified diffusion as

$$\frac{2\overline{\mu' s_{ij}}}{2\bar{\mu} S_{ij}} \sim \frac{v'}{\bar{v}} \frac{u/l}{U/L} = \left[\left(\frac{uL}{\bar{v}}\right)^{3a/4} - 1\right] \frac{u}{U} \frac{L}{l}, \tag{40}$$

so in principle $\overline{\mu' s_{ij}}$ cannot be neglected because $l/L \gg u/U$ (in fact, $l/L \sim (uL/\bar{v})^{-3/4}$).

Under certain conditions, however, it is possible to neglect $\overline{\mu' s_{ij}}$ as shown by Oliveira and Pinho [37]: basically, in 2D flows of shear rate ($\dot{\gamma}$) independent viscosity fluids the average strain rate $\bar{\dot{\varepsilon}}$ is zero, hence the fluctuating viscosity is an even function of $\dot{\varepsilon}'$. Thus, any correlation of μ' with other approximately Gaussian-distributed variable related to the small scale of turbulence will vanish because the odd moments of the normal distribution are zero [42]. So, no terms are dropped and the final form of the time-average momentum equation is that of Eq. (8).

5.2. The transport equations for $\overline{u_i u_j}$ and k

The transport equation for the Reynolds stress $\overline{u_i u_j}$ (Eq. (9)) contains several new molecular diffusion or dissipative terms. Their orders of magnitude are compared with that of the Newtonian-like dissipative term $2\bar{\mu}\overline{(\partial u_i/\partial x_k)(\partial u_j/\partial x_k)}$ which, note, is not equal to ε_{ij} due to the new definition of ε. The outcome of this analysis is contained in Table 1. At first sight, it is obvious that only the third and fourth terms in

Table 1
Order of magnitude relative to Newtonian dissipative term

Term	Order	Numerical
$\dfrac{\partial \bar{\mu}}{\partial x_k}\left(\dfrac{\overline{\partial u_i u_j}}{\partial x_k} + \dfrac{\overline{\partial u_k u_j}}{\partial x_i} + \dfrac{\overline{\partial u_k u_i}}{\partial x_j}\right)$	$\left(\dfrac{uL}{\bar{\nu}}\right)^{-3/2}$	1×10^{-6}
$-\dfrac{\partial \bar{\mu}}{\partial x_k}\overline{u_k s_{ij}}$	$\left(\dfrac{uL}{\bar{\nu}}\right)^{-3/4}$	1×10^{-3}
$\overline{\mu' \dfrac{\partial^2 u_i u_j}{\partial x_k \partial x_k}} - 2\overline{\mu' \dfrac{\partial u_i}{\partial x_k}\dfrac{\partial u_j}{\partial x_k}}$	$\left(\dfrac{uL}{\bar{\nu}}\right)^{3a/4} - 1$	1
$\overline{\dfrac{\partial \mu'}{\partial x_k}\dfrac{\partial u_i u_j}{\partial x_k}} + \overline{\dfrac{\partial \mu'}{\partial x_k}\left(u_j\dfrac{\partial u_k}{\partial x_i} + u_i\dfrac{\partial u_k}{\partial x_j}\right)}$	$\left(\dfrac{uL}{\bar{\nu}}\right)^{3a/4} - 1$	1
$\overline{\mu' u_j \dfrac{\partial^2 U_i}{\partial x_k \partial x_k}} + \overline{\mu' u_i \dfrac{\partial^2 U_j}{\partial x_k \partial x_k}}$	$\left[\left(\dfrac{uL}{\bar{\nu}}\right)^{3a/4} - 1\right]\dfrac{U}{u}\left(\dfrac{uL}{\bar{\nu}}\right)^{-3/2}$	1×10^{-5}
$\overline{u_j \dfrac{\partial \mu'}{\partial x_k}\left(\dfrac{\partial U_i}{\partial x_k} + \dfrac{\partial U_k}{\partial x_i}\right)} + \overline{u_i \dfrac{\partial \mu'}{\partial x_k}\left(\dfrac{\partial U_j}{\partial x_k} + \dfrac{\partial U_k}{\partial x_j}\right)}$	$\left[\left(\dfrac{uL}{\bar{\nu}}\right)^{3a/4} - 1\right]\dfrac{U}{u}\left(\dfrac{uL}{\bar{\nu}}\right)^{-3/4}$	1×10^{-3}

the table should be retained in a high Reynolds number formulation. Possibly, in a low Reynolds number formulation some of the other terms may be retained, but that requires a more extensive analysis left for the future as the present paper concentrates on modelling only a k–ε closure. Anyway, note also that the first term does not require modelling.

In conclusion, the simplified transport equation of $\overline{u_i u_j}$ is

$$\rho \frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} + \rho \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \rho \overline{u_i u_k}\frac{\partial U_j}{\partial x_k}$$
$$= -\rho\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} - \left(\frac{\partial}{\partial x_i}\overline{p' u_j} + \frac{\partial}{\partial x_j}\overline{p' u_i}\right) + \overline{p'\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)} + \bar{\mu}\frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} - 2\bar{\mu}\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$
$$+ \overline{\mu'\frac{\partial^2 u_i u_j}{\partial x_k \partial x_k}} - 2\overline{\mu'\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \overline{\frac{\partial \mu'}{\partial x_k}\frac{\partial u_i u_j}{\partial x_k}} + \overline{\frac{\partial \mu'}{\partial x_k}\left(u_j\frac{\partial u_k}{\partial x_i} + u_i\frac{\partial u_k}{\partial x_j}\right)}. \tag{41}$$

The transport equation of k (Eq. (10)), obtained by contraction of the indices of the equation of $\overline{u_i u_j}$, also contains new dissipative and diffusive terms and, as with the Reynolds stress equation, comparing their orders of magnitude with the order of magnitude of the dissipation term allows us to simplify it. The dissipative terms $2\overline{\mu' s_{ij}^2}$ and $2\bar{\mu}\overline{s_{ij}^2}$ are lumped together to define the dissipation ε (cf. Eq. (14)). The other dissipative term $2\overline{\mu' s_{ij} S_{ij}}$ is neglected in comparison to ε as in the Reynolds stress equation.

Terms $2\bar{\mu}\overline{u_i s_{ij}}$ and $2\overline{\mu' u_i s_{ij}}$ are components of the molecular diffusion of k which can be recast as $2\overline{(\bar{\mu} + \mu')u_i s_{ij}}$. Here, the diffusivity coefficient is the average molecular viscosity plus a contribution from the fluctuating molecular viscosity. This second contribution takes on positive and negative values and so it should be smaller than the former. The diffusion of k is usually very small, except at low Reynolds numbers and in the vicinity of walls, so the contribution from the fluctuating viscosity is in principle smaller and neglected by comparison, at least until further research shows that it should be kept,

i.e. as a first approximation one has

$$2\bar{\mu}\overline{u_i s_{ij}} + 2\overline{\mu' u_i s_{ij}} \approx 2\bar{\mu}\overline{u_i s_{ij}}. \tag{42}$$

Therefore, the simplified version of the transport equation of k is given by

$$\rho\frac{\mathrm{D}k}{\mathrm{D}t} = -\frac{\partial}{\partial x_j}\left[\overline{u_i p} + \frac{1}{2}\rho\overline{u_j u_i u_i} - 2\bar{\mu}\overline{u_i s_{ij}}\right] - \rho\varepsilon - \rho\overline{u_i u_j}S_{ij}. \tag{43}$$

5.3. Transport equation for ε

For the ε equation (Eq. (15)) the estimated order of magnitude of its various terms are summarised in Table 2.

Using terms Ia + Ib as reference, and considering l to be identical to the Kolmogorov length scale (leading to $l/L \sim (uL/\bar{v})^{-3/4}$), one ends up with the relative results of Table 3 which gives a better idea of the relevance of the various terms. To help in this determination, numerical values are given to a and to the two Reynolds numbers. For a there are three typical values of 0, -0.05 and $+0.05$, but in terms of order of magnitude $a = -0.05$ and $+0.05$ are equivalent. For the Reynolds numbers, a bulk Reynolds number (Re_U) of about 50,000 is considered and a turbulence intensity of about 10% gives $Re_u = 5000$. However, before proceeding, and to help in the critical assessment of the relevance of the various terms of the ε equation, the equivalent Newtonian equation is presented in its general form [43]:

$$\underbrace{\rho\frac{\partial\varepsilon}{\partial t}}_{\text{Ia}} + \underbrace{\rho U_i\frac{\partial\varepsilon}{\partial x_i}}_{\text{Ib}} = \underbrace{\mu\nabla^2\varepsilon}_{\text{IV}} - \underbrace{2\mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j}}\frac{\partial U_i}{\partial x_k}}_{\text{IIa}} - \underbrace{2\mu\overline{\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}}\frac{\partial U_i}{\partial x_k}}_{\text{IIb}} - \underbrace{2\mu\overline{u_k\frac{\partial u_i}{\partial x_j}}\frac{\partial^2 U_i}{\partial x_k\partial x_j}}_{\text{IIc}} - \underbrace{2\mu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}}}_{\text{IId}}$$

$$- \underbrace{\mu\frac{\partial}{\partial x_k}\left[\overline{u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right]}_{\text{IIIa}} - \underbrace{2\frac{\mu}{\rho}\frac{\partial}{\partial x_k}\left[\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_k}{\partial x_m}}\right]}_{\text{IIIb}} - \underbrace{2\frac{\mu^2}{\rho}\overline{\frac{\partial^2 u_i}{\partial x_k\partial x_m}\frac{\partial^2 u_i}{\partial x_k\partial x_m}}}_{\text{V}}. \tag{44}$$

In Eq. (44) the various terms are identified by the same codes used in the non-Newtonian equation (15) to facility comparisons, but caution must be exercised because the presence of molecular viscosity fluctuations can change their physical meaning as will be seen. Using now the numerical values of the previous paragraph, the numerical estimate of the relative order of magnitude of the terms in Table 3 leads to Table 4 and the following conclusions are drawn:

(i) For Newtonian fluids the purely non-Newtonian terms VI–XV vanish.
(ii) The terms that are common to the Newtonian and non-Newtonian equations (terms I–V), most of which have been modified, have the same order of magnitude regardless of the value of parameter a. The exceptions are the diffusive terms (turbulent diffusion: IIIa + IIIb; molecular diffusion: IV) which have been significantly modified and enhanced by the viscosity fluctuations. These terms were named by analogy to the Newtonian equation and considering the physics of the latter.
(iii) Of the new terms of the non-Newtonian equation, terms IX–XIV are irrelevant in comparison with terms VI + VII + VIII + XV. Terms IX–XII originate in the diffusive term $2\bar{\mu}S_{ij}$ of the momentum equation and terms XIII and XIV come from the other diffusive term $2\overline{\mu' s_{ij}}$. These two sets of

Table 2
Estimated order of magnitude of terms of ε (Eq. (15))

Term	Order
Ia + Ib	$\dfrac{Uu^3}{L^2}$
IIa + IIb	$\hat{v}\dfrac{Uu^2}{Ll^2}$
IIc	$\hat{v}\dfrac{Uu^2}{L^2l}$
IId	$\hat{v}\dfrac{u^3}{l^3}$
IIIa + IIIb	$\dfrac{u^3}{l^2}\left(\dfrac{\bar{v}}{L}+\dfrac{v'}{l}\right)$
IV	$\dfrac{u^2}{l^2}\left(\dfrac{v'^2}{l^2}+\dfrac{\bar{v}^2}{L^2}+\dfrac{v'\bar{v}}{l^2}\right)$
V	$\hat{v}^2\dfrac{u^2}{l^4}$
VI + VII + VIII	$\hat{v}v'\dfrac{u^2}{l^4}$
IX	$\hat{v}v'\dfrac{Uu}{Ll^3}$
X + XI	$\hat{v}v'\dfrac{Uu}{L^2l^2}$
XII	$\hat{v}v'\dfrac{Uu}{L^3l}$
XIIIa + XIIIb	$v'^2\dfrac{u^2}{Ll^3}$
XIV	$v'\dfrac{u^3}{Ll^2}$
XVa + XVb	$v'\dfrac{Uu^2}{l^3}$

terms have a negligible influence except perhaps in the perspective of a low Reynolds number flow and wall proximity. In this case terms IX and XIV look more important than the others but more detailed investigations are required to ascertain which should be kept under those conditions. Note that, with viscoelastic drag reducing fluids, a low Reynolds number formulation is essential because there is no universal law of the wall in contrast to what happens with Newtonian fluids. Since many of the arguments used in this order of magnitude analysis were formulated on the basis of high Reynolds number flow, some of the neglected terms may need to be re-evaluated close to walls. At this stage of knowledge on modelling turbulent viscoelastic flow we opted to neglect them.

(iv) Term XV is the most important of all terms as its order of magnitude is the highest. This does not make complete sense as the term should be balanced by at least another one. It is an indication that

Table 3
Estimated order of magnitude of terms of ε (Eq. (15)) relative to terms Ia + Ib[a]

Term	Order
IIa + IIb	$Re_u^{(2+3a)/4}$
IIc	$Re_u^{(3a-1)/4}$
IId	$Re_u^{(9+3a)/4} Re_U^{-1}$
IIIa + IIIb	$Re_u^{3/2} Re_U^{-1}[1 + Re_u^{(3+3a)/4} - Re_u^{3/4}]$
IV	$Re_u^{1/2} Re_U^{-1}\{1 + Re_u^{3/2}[Re_u^{3a/4} - 1]^2 + Re_u^{3/2}[Re_u^{3a/4} - 1]\}$
V	$Re_u^{(4+3a)/2} Re_U^{-1}$
VI + VII + VIII	$Re_u^{(8+3a)/4} Re_U^{-1}[Re_u^{3a/4} - 1]$
IX	$Re_u^{(1+3a)/4}[Re_u^{3a/4} - 1]$
X + XI	$Re_u^{(3a-2)/4}[Re_u^{3a/4} - 1]$
XII	$Re_u^{(3a-5)/4}[Re_u^{3a/4} - 1]$
XIIIa + XIIIb	$Re_u^{5/4} Re_U^{-1}[Re_u^{3a/4} - 1]^2$
XIV	$Re_u^{3/2} Re_U^{-1}[Re_u^{3a/4} - 1]$
XVa + XVb	$Re_u^{5/4}[Re_u^{3a/4} - 1]$

[a] Re_U is based on U and Re_u is based on u. In both cases the length scale is L and $\bar{\nu}$ is used.

perhaps the viscosity gradient should not be scaled with the length l as it was, but with L. In any case, the term is important and hence it is kept.

At this stage, and since the turbulent flows of viscoelastic solutions are almost always associated with Reynolds numbers well below those of Newtonian fluids, and given the need for a low Reynolds number

Table 4
Numerical estimate of the order of magnitude of terms of Eq. (15) relative to terms Ia + Ib

Term	Order
IIa + IIb	100
IIc	0.1
IId	1000
IIIa + IIIb	$a = 0 \to 1, a \neq 0 \to 1000$
IV	$a = 0 \to 0.001, a \neq 0 \to 100$
V	100
VI + VII + VIII	$a = 0 \to 0, a \neq 0 \to 100$
IX	$a = 0 \to 0, a \neq 0 \to 1$
X + XI	$a = 0 \to 0, a \neq 0 \to 0.001$
XII	$a = 0 \to 0, a \neq 0 \to 0.00001$
XIIIa + XIIIb	$a = 0 \to 0, a \neq 0 \to 0.1$
XIV	$a = 0 \to 0, a \neq 0 \to 1$
XVa + XVb	$a = 0 \to 0, a \neq 0 \to 10000$

formulation, the ε transport equation is

$$
\rho\frac{\partial \varepsilon}{\partial t} + \rho U_k \frac{\partial \varepsilon}{\partial x_k} = -2\frac{\partial U_i}{\partial x_k}\overline{\left[(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}\right]} - 2\frac{\partial U_k}{\partial x_m}\overline{\left[(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\right]}
$$

$$
- 2\frac{\partial^2 U_i}{\partial x_k \partial x_m}\overline{\left[(\bar{\mu}+\mu')u_k\frac{\partial u_i}{\partial x_m}\right]} - 2\overline{(\bar{\mu}+\mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}}
$$

$$
- \overline{(\bar{\mu}+\mu')\frac{\partial}{\partial x_k}\left[u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]} - 2\overline{(\bar{\nu}+\nu')\frac{\partial}{\partial x_i}\left[\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]}
$$

$$
+ \frac{\overline{(\bar{\mu}+\mu')^2}}{\rho}\frac{\partial}{\partial x_k}\overline{\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)\right]} - 2\frac{\overline{(\bar{\mu}+\mu')^2}}{\rho}\overline{\frac{\partial^2 u_i}{\partial x_k \partial x_m}\frac{\partial^2 u_i}{\partial x_k \partial x_m}}
$$

$$
+ 4\overline{(\bar{\nu}+\nu')\frac{\partial^2(\bar{\mu}+\mu')}{\partial x_m \partial x_k}\left[\frac{\partial u_i}{\partial x_m}s_{ik}\right]} + 4\overline{(\bar{\nu}+\nu')\frac{\partial(\bar{\mu}+\mu')}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial s_{ik}}{\partial x_m}}
$$

$$
+ 2\overline{(\bar{\nu}+\nu')\frac{\partial(\bar{\mu}+\mu')}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)} + 4S_{ik}\overline{\left[(\bar{\nu}+\nu')\frac{\partial u_i}{\partial x_m}\frac{\partial^2\mu'}{\partial x_m \partial x_k}\right]}
$$

$$
- 4\overline{\nu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial\mu'}{\partial x_k}s_{ik}\right]} - 2\overline{\nu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\mu'\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right]} + 2\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \overline{u_i u_k}}{\partial x_k}\right]}
$$

$$
+ \frac{\overline{\partial(\bar{\mu}+\mu')}}{\partial t}\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m} + U_k\frac{\overline{\partial(\bar{\mu}+\mu')}}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}. \tag{45}
$$

This equation will be further simplified when discussing its modelling in the perspective of a k–ε closure, in the next section.

6. Modelling the transport equations for a k–ε closure

The analysis of order of magnitude of the previous section has shown what terms of the transport equations must be retained and which to neglect under some assumptions. The relevant terms are of two types: those that can be directly evaluated, such as the molecular diffusion or the production of turbulence in the k equation, and those that must be modelled, such as the turbulent diffusion.

6.1. Momentum equation

The aim is to solve the momentum equation (Eq. (8)) which possesses three terms that require modelling: the mean molecular stress $2\bar{\mu}S_{ij}$, the molecular turbulent stress $2\overline{\mu's_{ij}}$ and the Reynolds stress $\overline{u_i u_j}$.

For the mean molecular term, closure is ensured by the expression for the average viscosity (Eq. (35)).

The molecular-turbulent stress is a new term coupling the fluctuations in viscosity and in the rate of strain tensor. It is a relevant term for shear-thinning fluids, that can only be neglected in non- or weakly shear-thinning fluids in 2D mean flows, such as in boundary layers, jets or pipe flows. As mentioned in Section 3, $2\overline{\mu's_{ij}}$ brings into the momentum equation of the polymer solution both viscous and elastic

contributions from the polymer molecules. However, $2\bar{\mu}S_{ij}$ also include viscous and elastic contributions from the polymer molecules in addition to he viscous contribution from the Newtonian solvent. It is difficult to ascertain how the polymer contributions are split between both terms, but the fact that $2\overline{\mu's_{ij}}$ becomes negligible under certain conditions suggests that $2\bar{\mu}S_{ij}$ takes in a significant amount of the effect. At present $2\overline{\mu's_{ij}}$ is dropped and its effect is basically taken by $2\bar{\mu}S_{ij}$ and a new damping function f_v to be introduced in the follow-up paper [1]. In the near future, this must be improved by an adequate modelling of $2\overline{\mu's_{ij}}$.

Finally, the Reynolds stress requires modelling and henceforth in this work a first-order turbulence closure of the k–ε type is the choice. The selection of such model may seem too simplistic an approach but the truth is that there is no single-point turbulence model for drag reducing fluids that combine the effects of turbulence and non-linear rheology. Therefore, a k–ε formulation seems adequate as a starting point for semi-quantitative predictions meaning that the trends will be captured, a significant drag reduction will also obtained but predictions will not always match experimental results.

For the Reynolds stress, the Boussinesq approximation or gradient hypothesis is used

$$-\overline{u_i u_j} = v_{\mathrm{T}} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}, \tag{46}$$

with the turbulent eddy viscosity given by the Prandtl–Kolmogorov equation

$$v_{\mathrm{T}} = C_\mu f_\mu \frac{k^2}{\varepsilon}. \tag{47}$$

In Eq. (47) there is a damping function f_μ that is needed for low Reynolds number models, but is equal to 1 in high Reynolds number formulations and away from the wall. This, and other damping functions appearing later, account for physical inadequacies in modelling [44]. Naturally, since the fluids involved are now non-Newtonian, the damping function f_μ, parameter C_μ and other parameters and damping functions found in Newtonian models must be evaluated differently. This is done by Cruz and Pinho [1] and is partially inspired by the work of Cruz et al. [31] for inelastic power law fluids.

6.2. Transport equation for k

In Eq. (43) only the terms within the square brackets need modelling. There is one term due to the inter-action of velocity and pressure fluctuations (pressure diffusion) and a second term due to the interaction of velocity fluctuations with Reynolds stresses, both grouped together under the name of turbulent diffu-sion. The third term is the molecular diffusion of turbulent kinetic energy differing from the Newtonian definition because it involves the average molecular viscosity.

Pressure diffusion has a small contribution in Newtonian flows and is not expected to behave differently with the non-Newtonian fluids. In fact, the DNS simulations of Dimitropoulos et al. [22] and De Angelis et al. [38] show that viscoelasticity decreases pressure diffusion. Hence, the usual approach in its modelling is used: pressure diffusion is lumped with turbulent diffusion and modelled as a symmetric term or, equivalently it is neglected.

Since turbulent diffusion is independent of the constitutive equation, it is modelled exactly as for Newtonian fluids using the classical gradient model

$$\overline{u_j p} + \frac{1}{2} \overline{u_i u_j u_i} = -\rho \frac{v_{\mathrm{T}}}{\sigma_k} \frac{\partial k}{\partial x_j}, \tag{48}$$

where σ_k is an empirical coefficient called the turbulent Prandtl number. The final form of the transport equation of k is

$$\rho \frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_j}\left[\rho\frac{\nu_{\mathrm{T}}}{\sigma_k}\frac{\partial k}{\partial x_j} + \bar{\mu}\frac{\partial k}{\partial x_j}\right] - \rho\overline{u_i u_j}S_{ij} - \rho\varepsilon. \tag{49}$$

6.3. Transport equation for ε

The equation with the most substantial number of modifications is the transport equation for the rate of dissipation of turbulent kinetic energy. Unfortunately, there is still no viscoelastic DNS work with a budget of ε to guide modelling.

The dissipation equation concerns physical processes in the dissipative range but for its modelling with Newtonian fluids ε is viewed rather as an energy flow rate in the energy cascade, i.e. a large scale motion quantity. Consequently, the modelled transport equation of ε for Newtonian fluids is basically empirical and the same approach is used in this work. Therefore, inspection of the various terms of the exact equation of ε basically serves the purpose of identifying similarities with the corresponding Newtonian equation and the physical mechanisms involved in order to help in their modelling.

Terms IIa, IIb and IIc correspond to the production of ε, because they quantify the interaction between ε and the mean flow gradient, and now they also include the effect of variable viscosity. Terms IId in Eqs. (15) and (44) pertain to the generation of vorticity fluctuations which are often included as part of the destruction of ε [45]. In the Newtonian equation, term IIc is usually neglected in comparison with term IId with the argument that the fluctuations of velocity and its gradient in IIc are less well correlated than the gradient quantities in IId [45]. The same approach is adopted here and it is worth remembering that in the presence of drag reducing fluids the correlations between turbulent quantities usually decrease in comparison with their Newtonian equivalents. This was seen to be the case in the Reynolds stress and k budgets obtained by Dimitropoulos et al. [22] and De Angelis et al. [38] with DNS and experimentally in several works [13], so there is no reason to believe that it can not be so also for the ε equation.

Regarding terms IIa and IIb, under the assumption of isotropic dissipation (small scales) they are neglected here. This is one of the issues that may have to be reviewed in the future given the tendency for fluid elasticity to accentuate anisotropy of turbulence (large scales).

$$\overline{\mu\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_m}} = 0, \quad \text{when } i = j \text{ and } l \neq m, \; i \neq j \text{ and } l = m. \tag{50}$$

For Newtonian fluids, Hanjalic and Launder [46] modelled the production of ε (terms IIa + IIb) by not assuming isotropy of dissipation but then modelled the generation of vorticity fluctuations (IId) and the destruction of ε (V) in a different way so that, the sum IIa + IIb + IIc + V gives rise to two terms in the modelled equation that exactly match the modelling of other authors who assumed isotropy of dissipation (Eq. (50)). Therefore, defining the production of ε as $P_\varepsilon \equiv$ IIa + IIb + IIc + IId, this is modelled as

$$P_\varepsilon \equiv -\rho f_1 C_{\varepsilon 1}\frac{\varepsilon}{k}\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} = \rho f_1 C_{\varepsilon 1}\frac{\varepsilon}{k}\nu_{\mathrm{T}}\left(\frac{\partial U_i}{\partial x_j}\right)^2 = \rho f_1 C_{\varepsilon 1}\frac{\varepsilon}{k}C_\mu f_\mu\frac{k^2}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j}\right)^2. \tag{51}$$

This model of production automatically considers the modifications due to the variable molecular viscosity in the modified definition of average rate of dissipation. The damping function f_1 accounts for

modelling inadequacies at low Reynolds numbers and near walls may need to be modified to account for non-Newtonian effects.

The next two terms in Eqs. (15) and (44) (terms IIIa + IIIb) must be analysed cautiously. In the Newtonian equation (44) IIIa represents the turbulent diffusion of dissipation by the velocity fluctuations, whereas the role of pressure fluctuations on turbulent diffusion is accounted for by term IIIb. For the non-Newtonian fluids it is convenient to further manipulate these terms. Term IIIa is separated into two subterms as follows:

$$\underbrace{-\overline{(\bar{\mu}+\mu')\frac{\partial}{\partial x_k}\left[u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]}}_{\text{IIIa}} = \underbrace{-\frac{\partial}{\partial x_k}\left(\rho\overline{u_k\hat{\varepsilon}}\right)}_{\text{IIIa1}} + \underbrace{\overline{u_k\frac{\partial(\bar{\mu}+\mu')}{\partial x_k}\left(\frac{\partial u_i}{\partial x_m}\right)^2}}_{\text{IIIa2}}. \tag{52}$$

The first term on the right-hand side is the classical turbulent diffusion and the second term is purely non-Newtonian, having the same order of magnitude of the whole term IIIa in Table 2. This is very important because the order of magnitude of term IIIa for non-Newtonian fluids was 100 times larger than for Newtonian fluids, i.e. the magnitude of IIIa1 is quite small in comparison to that of term IIIa2. Modelling of IIIa2 is discussed below in Eq. (59).

Term IIIb can also be split as

$$\underbrace{-\overline{(\bar{\nu}+\nu')\frac{\partial}{\partial x_k}\left[\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]}}_{\text{IIIb}} = \underbrace{-2\frac{\partial}{\partial x_k}\overline{\left[(\bar{\nu}+\nu')\frac{\partial u_i}{\partial x_m}\frac{\partial p'}{\partial x_m}\right]}}_{\text{IIIb1}} + \underbrace{2\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial(\bar{\mu}+\mu')}{\partial x_i}}}_{\text{IIIb2}}, \tag{53}$$

with term IIIb1 accounting for diffusion by pressure fluctuations and the second term associated with viscosity variations. The latter is again the predominant contribution and is the main responsible for the order of magnitude of the whole term IIIb.

However, at the moment term IIIb2 is neglected for lack of knowledge on how to model it. Politis [30] has considered this term to have a negligible influence and assumed that its magnitude was smaller and similar to that of our term IIIb1. To estimate the order of magnitude of term IIIb2, he used L as the length scale for the gradient of pressure fluctuations (we used l). In the absence of more information, we consider the view of Politis [30] and so the effect of coupling pressure gradient with viscosity gradient fluctuations is assumed included in the modelling of term IIIb1. In conclusion, terms IIIa1 + IIIb1 + IIIb2 are modelled together and as part of turbulent diffusion, but in the future term IIIb2 may have to be specified separately and in a different way.

Finally, there is an extra non-Newtonian turbulent diffusion term due to viscosity fluctuations (term XIV in Eq. (15)). As seen above, term XIV is small compared with terms IIIa1 + IIIb1 and so, together these three contributions, in addition to term IIIb2, will be denoted as D_ε. In the analysis of Politis [30] for purely viscous fluids term XIV was also shown to be negligible.

In a classical k–ε closure, the turbulent diffusion D_ε is often modelled with a gradient transport hypothesis: the argument is that, for a continuum flow, the time and length scales of the fluctuations and of the molecular processes are different by many orders of magnitude, but the time and length scales of the mean and fluctuating flows are of similar magnitudes. For the k–ε closure it is further assumed that the turbulence is isotropic leading to

$$D_\varepsilon = \frac{\partial}{\partial x_i}\left(\rho C_\varepsilon \frac{k^2}{\varepsilon}\frac{\partial\varepsilon}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\rho\frac{\nu_\text{T}}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_i}\right). \tag{54}$$

There are also alternative models to account for anisotropic turbulent diffusion of ε and these will be needed in the future, since turbulence anisotropy is enhanced in viscoelastic turbulent flow.

Molecular diffusion of ε is a rather complex combination of terms in the non-Newtonian equation. Whereas in Eq. (44) that mechanism is only represented by term IV ($\mu \nabla^2 \varepsilon$), in Eq. (15) molecular diffusion is IV + VI + VII + VIII. At high Reynolds number flows the molecular diffusion is usually neglected in comparison to turbulent diffusion, but in a low Reynolds number formulation it must be included. Although one may assume that the contributions VI, VII and VIII can be neglected, because they involve first and second derivatives of the total viscosity, the order of magnitude analysis has shown that they tend to be as important as term IV for a non-Newtonian fluid. However, their relevance is well below that of the new terms XV and IIIa2, hence, as a first approximation those new terms (VI, VII and VIII) are not considered.

Therefore, molecular diffusion of dissipation is calculated as

$$\frac{\partial}{\partial x_i} \left(\bar{\mu} \frac{\partial \varepsilon}{\partial x_j} \right), \tag{55}$$

where the non-Newtonian contribution to the viscosity is accounted for in $\bar{\mu}$.

The last term on the right-hand side of the Newtonian equation (44) represents the destruction of dissipation by viscosity (Φ_ε). Its equivalent form for non-Newtonian fluids is term V in Eq. (15). In the standard model for Newtonian fluids, Hanjalic and Launder [46] consider

$$\Phi_\varepsilon = -\rho f_2 C_{\varepsilon 2} \frac{\varepsilon^2}{k}, \tag{56}$$

where f_2 takes the value of 1. Near the wall this damping function accounts for deficiencies in modelling Φ_ε. Here, term V of the non-Newtonian equation (15) is modelled in the same way, i.e. by Eq. (56) and the damping function f_2 may have to be modified in the future.

Concerning the most important new term (XVa + XVb), Politis [30] has also shown it to be important in his purely viscous analysis, and his modelling options are taken here. Term XVa is modelled as

$$\rho \overline{\frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_m} \frac{\partial (\bar{\mu} + \mu')}{\partial t}} = \rho \overline{\frac{\hat{\varepsilon}}{\hat{\mu}} \frac{\partial \hat{\mu}}{\partial t}} \approx \rho C_{\varepsilon 3} \frac{\varepsilon}{\bar{\mu}} \frac{\partial \bar{\mu}}{\partial t}, \tag{57}$$

but is only required for transient flows. Similarly, for term XVb

$$\rho U_k \overline{\frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_m} \frac{\partial (\bar{\mu} + \mu')}{\partial x_k}} = \rho U_k \overline{\frac{\hat{\varepsilon}}{\hat{\mu}} \frac{\partial \hat{\mu}}{\partial x_k}} \approx \rho C_{\varepsilon 3} \frac{\varepsilon}{\bar{\mu}} U_k \frac{\partial \bar{\mu}}{\partial x_k}. \tag{58}$$

The relevance of these two terms XV is not surprising: they originate in the convective momentum-transport and take into account the gradients of viscosity. In fact, the advection of ε involves the gradients of ε and so, with its new definition it is necessary to account for both the spatial gradients of the instantaneous rates of deformation, this is the classical advective term of ε, as well as the spatial gradients of the variable viscosity which leads to the new term of Eq. (58). For the time gradient part of ε a similar decomposition is required leading to the new term of Eq. (57). The two classical terms derived from the convective momentum transport, and accounting for both the time and spatial gradients of the rates of deformation, are those on the left-hand side of Eq. (15), the convective transport of ε (terms Ia and Ib).

Naturally, under conditions of symmetry for which the convective terms Ia and Ib are null, the viscosity gradient convective terms XVa and XVb also become zero.

In modelling XVa and XVb the gradient of instantaneous viscosity is approximated by the gradient of average viscosity and then this quantity is decoupled from the square gradients of velocity. Then, this square gradient of velocity is related to the rate of dissipation divided by the time-average viscosity and a parameter $C_{\varepsilon 3}$ is introduced to account for deficiencies in this decoupling. Parameter $C_{\varepsilon 3}$ will require quantification in the future. Note that in the second part of this work on turbulent fully developed pipe flow [1] this term does not appear due to geometric symmetry.

Term IIIa2 has mathematical similarities to term XVb, in that the fluctuating velocity has substituted the mean velocity. Physically, the term represents turbulent diffusion (cf. Eq. (52)) and hence it will be modelled as

$$\overline{u_k \frac{\partial(\bar{\mu} + \mu')}{\partial x_k} \frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_m}} = \overline{u_k \frac{\rho \hat{\varepsilon}}{(\bar{\mu} + \mu')} \frac{\partial(\bar{\mu} + \mu')}{\partial x_k}} \sim C_{\varepsilon 4} \rho \overline{u_k \hat{\varepsilon}} \frac{1}{\bar{\mu}} \frac{\partial \bar{\mu}}{\partial x_k} \sim C_{\varepsilon 4} \frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon \bar{\nu}} \frac{\partial \varepsilon}{\partial x_k} \frac{\partial \bar{\mu}}{\partial x_k}, \tag{59}$$

where use is made of the gradient transport hypothesis as in term IIIa1:

$$\overline{u_k \hat{\varepsilon}} \sim \frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_k} \tag{60}$$

and in Eq. (59) some of the approximations are the same used to model XVb. Parameter $C_{\varepsilon 4}$ requires quantification.

In conclusion, the final form of the modelled dissipation equation for a low Reynolds number formulation is

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho U_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\left(\bar{\mu} + \rho \frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_i} \right] + \rho f_1 C_{\varepsilon 1} \frac{\varepsilon}{k} C_\mu \frac{k^2}{\varepsilon} \left(\frac{\partial U_i}{\partial x_j} \right)^2 - \rho f_2 C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \rho C_{\varepsilon 3} \frac{\varepsilon}{\bar{\mu}} \frac{\partial \bar{\mu}}{\partial t}$$
$$+ \rho C_{\varepsilon 3} \frac{\varepsilon}{\bar{\mu}} U_i \frac{\partial \bar{\mu}}{\partial x_i} + C_{\varepsilon 4} \frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon \bar{\nu}} \frac{\partial \varepsilon}{\partial x_k} \frac{\partial \bar{\mu}}{\partial x_k}. \tag{61}$$

6.4. Model parameters and functions

The turbulence model has introduced several parameters and damping functions: most also exist in a Newtonian formulation ($C_\mu, f_\mu, f_1, f_2, C_{\varepsilon 1}, C_{\varepsilon 2}$), whereas others are new and specific to the non-Newtonian formulation ($C_{\varepsilon 3}, A_\varepsilon, A_2, C_{\varepsilon 4}$). This does not mean that the first set of parameters and functions is known and only the second must be quantified. The non-Newtonian rheology certainly changes the turbulence dynamics typical of Newtonian fluids, so classical terms are also affected. This means that there are changes in the turbulent flow behaviour of viscoelastic fluids in reference flows that are essential to determine the numerical values of the parameters as we see in the next section. Unfortunately, very often no such data are available and the Newtonian values are kept. This issue, together with other relevant points, are addressed in the second part of this work [1] which applies this turbulence model to the prediction of fully developed pipe flow of different polymer solutions and compares the results with experimental data.

7. Behaviour in isotropic grid turbulence

One of the fundamental flows for determining turbulence model parameters is the decay of isotropic turbulence generated in a grid [44,45]. Such simple flows can also show whether new formulations are able to capture certain essential features.

Given the properties of this flow and of isotropic turbulence, and particularly the absence of directional correlations for triple correlations of vectors and for double correlations between vectors and scalars, the transport equations of k and ε take the forms of Eqs. (62) and (63), respectively:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = U_0 \frac{\mathrm{d}k}{\mathrm{d}x} = -\varepsilon, \tag{62}$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = U_0 \frac{\mathrm{d}\varepsilon}{\mathrm{d}x} = -C_{\varepsilon 2} \frac{\varepsilon^2}{k} + C_{\varepsilon 3} \frac{\varepsilon}{\bar{\mu}} U_0 \frac{\mathrm{d}\bar{\mu}}{\mathrm{d}x}, \tag{63}$$

where U_0 is the uniform flow mean velocity in the x direction. The k equation is the same regardless of fluid rheology, whereas the last term on the right-hand side of Eq. (63) is absent for a Newtonian fluid. However, we also assume that $C_{\varepsilon 2}$ takes on the same values for Newtonian and non-Newtonian fluids.

According to the adopted constitutive model, where the influence of the polymer is basically introduced in the non-linear viscosity variation, and more specifically extensional effects are introduced via exponent p, it is the spatial gradient of the molecular viscosity that will determine whether the rate of decay of k is faster or slower than for Newtonian fluids. Before proceeding it is helpful to understand the physical meaning of this new term in Eq. (63). As mentioned when the new term was modelled in the previous section (see discussion below Eq. (58)), it is not too different from the term on the left-hand side of Eq. (63): convective terms account for spatial gradients of ε, but now it is necessary to account for both the spatial gradients of the product of instantaneous rates of deformation (leading to the classical left-hand side term) as well as to the gradients of viscosity, leading to the new term in Eq. (63). So, the new term can be thought of as a correction to the classical convective term because of the slightly different definition of ε taking into account a variable viscosity.

In the following analysis of the rate of decay of k we assume that the production of turbulence at the grid is independent of fluid rheology, i.e. the initial values of k and ε are identical for Newtonian and non-Newtonian fluids. It is also helpful to analyse first the Newtonian situation for which it is easy to arrive at an expression for k and ε [45]. Since there is no production of turbulence, k and ε must decay longitudinally according to

$$k = ax^r, \tag{64}$$

$$\varepsilon = -U_0 arx^{r-1}, \tag{65}$$

with r taking some negative value which for Newtonian fluids is close to -1.1. This is a consequence of the first term on the right-hand side of Eq. (63) being always negative.

For drag reducing fluids, Eqs. (64) and (65) are also solutions to Eqs. (62) and (63) although the complex dependence of $\bar{\mu}$ on ε and k (Eq. (35)) make it more laborious and the parameters a and r are necessarily different from the corresponding Newtonian parameters. This is obvious because the absence of turbulence production requires k to decay. The real issue, however, is whether this decay is faster or slower than for Newtonian fluids.

Since this flow has null average shear rates, in the absence of turbulence the molecular viscosity will be determined by the first Newtonian plateau. Turbulence will change the molecular viscosity and here there are two possibilities:

(i) If parameters n and p are such that $\bar{\mu}$ shear-thins (say, $n < 1$, $p = 1$), then $\mathrm{d}\bar{\mu}/\mathrm{d}x > 0$ because of the decrease of k with x. This entails a positive second term on the right-hand side of Eq. (63) and so $0 > (\mathrm{d}\varepsilon/\mathrm{d}x)_{\mathrm{NN}} > (\mathrm{d}\varepsilon/\mathrm{d}x)_{\mathrm{N}}$, i.e. the rate of dissipation decays slower for non-Newtonian than for Newtonian fluids (subscripts N and NN stand for Newtonian and non-Newtonian fluids, respectively). The consequence is that the non-Newtonian $\mathrm{d}k/\mathrm{d}x$ is more negative than the Newtonian $\mathrm{d}k/\mathrm{d}x$ and so $r_{\mathrm{NN}} < r_{\mathrm{N}} < 0$ in Eqs. (64) and (65).

(ii) If parameters n and p are such that $\bar{\mu}$ strain-hardens (say, $n = 1$, $p > 1$), then $\mathrm{d}\bar{\mu}/\mathrm{d}x < 0$ again because of the decrease of k with x. Now, in contrast to the previous case the second term on the right-hand side of Eq. (63) is negative, therefore the rate of dissipation decays faster $0 > (\mathrm{d}\varepsilon/\mathrm{d}x)_{\mathrm{N}} > (\mathrm{d}\varepsilon/\mathrm{d}x)_{\mathrm{NN}}$, $\varepsilon_{\mathrm{NN}} < \varepsilon_{\mathrm{N}}$ and the turbulent kinetic energy for the non-Newtonian fluid decays slower than for the Newtonian case.

It is this second case ($p > 1$, $n = 1$) that is representative of many dilute polymer solutions that show drag reduction in pipe flow. Unfortunately, the data in the literature for the decay of grid turbulence are scarce but still show tendencies of behaviour. Greated [48] measured a Newtonian decay of k with $r = -1.9$, when it should be closer to -1.1, but his measurements indicated a non-Newtonian exponent of -1.5 for 1000 wppm polyethylene oxide, i.e. his rate of decay of turbulence for drag reducing fluids was lower than for Newtonian fluids. This reduction in the decay rate of k for polymer solutions was later confirmed by Mc Comb et al. [49].

Note that in this work the determination of parameters K_{v}, K_{e}, n and p in Eq. (7) has not been specified. As shown in the companion paper [1] the term with K_{v} and n accounts for the pure viscometric behaviour, whereas the term involving K_{e} and p quantifies the Trouton ratio, meaning that $p > 1$ only for fluids where the ratio of extensional over the shear viscosities increase with the rate of deformation of the fluid. These are the fluids that can have large amounts of drag reductions (see also [37]).

8. Conclusions

The transport equations for momentum, Reynolds stresses, turbulent kinetic energy and its rate of dissipation were derived for the GNF model. This model accurately predicts shear-thinning, but was modified to include strain rate effects aimed at mimicking strain-hardening of the extensional viscosity, considered to be the most important rheological characteristic for the occurrence of drag reduction. These transport equations, except that for the Reynolds stresses, were subsequently simplified after an analysis of order of magnitude identified the relevant term to retain. Since viscosity is a non-linear function of kinematic quantities, it will have fluctuations due to turbulence, and in this work a closed form for the average molecular viscosity was also derived.

At the end of the paper a low Reynolds number k–ε model is formulated, which includes some new terms not present in the classical k–ε model for Newtonian fluids. The suggested model is presented in closed form and the remaining details required for its useful application to engineering wall flows, such as the quantification of model parameters and damping functions, are presented in a follow-up paper [1]. It was also shown that the present k–ε model can predict a slower rate of decay of grid generated turbulence

than that of Newtonian fluids for fluids having a Trouton-thickening viscosity variation, in agreement with results from the literature.

The results of this work are not restricted to the model to be used in [1], but remain valid for other specific turbulence models for viscoelastic fluids, provided their rheology is modelled with a GNF equation. It is realised that further improvements to the model derived at the end are still required and that more advanced first-order non-linear and second-order models are advantageous. However, we should first fully explore the current model to weigh its merits and shortcomings, the subject of the paper by Cruz and Pinho [1].

Acknowledgements

Appendix A. Conservation equations for generalised Newtonian fluids in turbulent flow

The derivation of the various conservation equations for the turbulent flow of GNF fluids follows the same guidelines as for Newtonian fluids.

The continuity equation remains unchanged as it is independent of the rheological constitutive equation. However, the momentum equation, and all other dependent conservation equations will be affected. The starting point is always the Cauchy equation for the *i*-component of the velocity vector.

In all that follows a hat designates instantaneous values, capital letters or overbars refer to average values and small letters or primes refer to fluctuations.

A.1. Momentum equation

The Cauchy equation for the *i*-component of the instantaneous velocity vector is

$$\rho \frac{D\hat{U}_i}{Dt} = -\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial \hat{\sigma}_{ik}}{\partial x_k}, \tag{A.1}$$

where σ_{ik} is the deviatoric stress tensor and D/Dt represents the total derivative ($D/Dt \equiv \partial/\partial t + U_k \partial/\partial x_k$).

Applying the Reynolds decomposition and taking the time-average, after substitution of the constitutive equation for the stress tensor, results in

$$\rho \frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial(2\bar{\mu}S_{ik} + 2\overline{\mu' s_{ik}} - \rho\overline{u_i u_k})}{\partial x_k}. \tag{A.2}$$

Eq. (A.2) is the momentum conservation equation for GNF fluids and it has an extra term ($2\overline{\mu' s_{ik}}$) relative to the corresponding Newtonian equation.

A.2. The Reynolds stress transport equation

Following Hinze [47], to deduce the transport equation for the Reynolds stress component *ij*, one first takes the instantaneous *i*-component momentum equation (A.3):

$$\rho \frac{\partial(U_i + u_i)}{\partial t} + \rho(U_k + u_k)\frac{\partial(U_i + u_i)}{\partial x_k} = -\frac{\partial p}{\partial x_i} - \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_k}[2(\bar{\mu} + \mu')(S_{ik} + s_{ik})], \tag{A.3}$$

and subtracts from it the time-averaged momentum conservation (Eq. (A.2)) to yield an equation on the fluctuations of the *i*-velocity component (Lu_i) onto which the zero quantity $\rho u_i(\partial u_k/\partial x_k)$ was added.

$$Lu_i: \quad \rho\frac{\partial u_i}{\partial t} + \rho U_k \frac{\partial u_i}{\partial x_k} + \rho u_k \frac{\partial U_i}{\partial x_k} + \rho \frac{\partial}{\partial x_k}(u_i u_k - \overline{u_i u_k})$$
$$= -\frac{\partial p'}{\partial x_i} + 2\frac{\partial}{\partial x_k}(\bar{\mu}s_{ik} + \mu' S_{ik} + \mu' s_{ik} - \overline{\mu' s_{ik}}). \tag{A.4}$$

Identically for the *j*-component of momentum gives equation Lu_j.

Next, equations Lu_i and Lu_j are combined as $u_i Lu_j + u_j Lu_i$ to yield Eq. (A.5)

$$\rho \frac{\partial(u_i u_j)}{\partial t} + \rho U_k \left(u_j \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial u_j}{\partial x_k} \right) + \rho u_j u_k \frac{\partial U_i}{\partial x_k} + \rho u_i u_k \frac{\partial U_j}{\partial x_k}$$
$$+ \rho u_j \frac{\partial}{\partial x_k}(u_i u_k - \overline{u_i u_k}) + \rho u_i \frac{\partial}{\partial x_k}(u_j u_k - \overline{u_j u_k})$$
$$= -u_j \frac{\partial p'}{\partial x_i} - u_i \frac{\partial p'}{\partial x_j} + 2u_j \frac{\partial}{\partial x_k}(\bar{\mu}s_{ik} + \mu' s_{ik} + \mu' S_{ik} - \overline{\mu' s_{ik}})$$
$$+ 2u_i \frac{\partial}{\partial x_k}(\bar{\mu}s_{jk} + \mu' s_{jk} + \mu' S_{jk} - \overline{\mu' s_{jk}}). \tag{A.5}$$

Time-averaging equation (A.5), grouping together the advective terms and after some algebra one gets

$$\rho \frac{D\overline{u_i u_j}}{Dt} + \rho \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \rho \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \rho \frac{\partial}{\partial x_k}\overline{u_i u_j u_k}$$
$$= -\frac{\partial}{\partial x_i}\overline{p'u_j} - \frac{\partial}{\partial x_j}\overline{p'u_i} + \overline{p'\frac{\partial u_j}{\partial x_i}} + \overline{p'\frac{\partial u_i}{\partial x_j}} + \overline{2u_j \frac{\partial}{\partial x_k}(\bar{\mu}s_{ik})} + \overline{2u_i \frac{\partial}{\partial x_k}(\bar{\mu}s_{jk})}$$
$$+ \overline{2u_j \frac{\partial}{\partial x_k}(\mu' s_{ik})} + \overline{2u_i \frac{\partial}{\partial x_k}(\mu' s_{jk})} + \overline{2u_j \frac{\partial}{\partial x_k}(\mu' S_{ik})} + \overline{2u_i \frac{\partial}{\partial x_k}(\mu' S_{jk})}. \tag{A.6}$$

All the non-viscous terms were written in the traditional way and below the viscous terms are rearranged to facilitate their identification. Eq. (A.6) has "Newtonian-like" terms (some of those involving the average viscosity $\bar{\mu}$) and non-Newtonian terms (some of those involving $\bar{\mu}$ and all terms with μ'). Looking first at the average viscosity terms

$$\overline{2u_j \frac{\partial}{\partial x_k}(\bar{\mu}s_{ik})} + \overline{2u_i \frac{\partial}{\partial x_k}(\bar{\mu}s_{jk})} = \overline{\bar{\mu}u_j \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} + \overline{\bar{\mu}u_i \frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)}$$
$$+ \overline{\frac{\partial\bar{\mu}}{\partial x_k}u_j\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} + \overline{\frac{\partial\bar{\mu}}{\partial x_k}u_i\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)}. \tag{A.7}$$

Since

$$\frac{\partial^2}{\partial x_k \partial x_k}\overline{u_i u_j} = \overline{u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k}} + \overline{u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k}} + 2\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \quad \text{and} \quad \frac{\partial}{\partial x_k}\left(\frac{\partial u_k}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = 0,$$

the right-hand side terms in Eq. (A.7) become

$$\overline{\bar{\mu}u_j\frac{\partial^2 u_i}{\partial x_k \partial x_k}} + \overline{\bar{\mu}u_i\frac{\partial^2 u_j}{\partial x_k \partial x_k}} + \frac{\partial\bar{\mu}}{\partial x_k}\frac{\partial}{\partial x_k}(\overline{u_i u_j}) + \frac{\partial\bar{\mu}}{\partial x_k}\left[\overline{u_j\frac{\partial u_k}{\partial x_i}} + \overline{u_i\frac{\partial u_k}{\partial x_j}}\right]$$

$$= \bar{\mu}\frac{\partial^2}{\partial x_k \partial x_k}\overline{u_i u_j} - 2\bar{\mu}\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \frac{\partial\bar{\mu}}{\partial x_k}\frac{\partial}{\partial x_k}\overline{u_i u_j} + \frac{\partial\bar{\mu}}{\partial x_k}\left[\frac{\partial}{\partial x_i}\overline{u_k u_j} + \frac{\partial}{\partial x_j}\overline{u_k u_i} - 2\overline{u_k s_{ij}}\right]. \qquad (A.8)$$

In Eq. (A.8) the first two terms are "Newtonian-like", whereas the last two terms are non-Newtonian and take into account the gradients of the average molecular viscosity within the flow field.

A similar rearrangement can be carried out for the other two groups of two terms. First, for those involving fluctuations of the viscosity and of the rate of deformation tensor

$$\overline{2u_j\frac{\partial}{\partial x_k}(\mu' s_{ik})} + \overline{2u_i\frac{\partial}{\partial x_k}(\mu' s_{jk})}$$

$$= \overline{\mu'u_j\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} + \overline{\mu'u_i\frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)}$$

$$+ \overline{\frac{\partial\mu'}{\partial x_k}u_j\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} + \overline{\frac{\partial\mu'}{\partial x_k}u_i\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)}$$

$$= \overline{\mu'u_j\frac{\partial^2 u_i}{\partial x_k \partial x_k}} + \overline{\mu'u_i\frac{\partial^2 u_j}{\partial x_k \partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\frac{\partial u_i u_j}{\partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\left(u_j\frac{\partial u_k}{\partial x_i} + u_i\frac{\partial u_k}{\partial x_j}\right)}$$

$$= \overline{\mu'\frac{\partial^2 u_i u_j}{\partial x_k \partial x_k}} - 2\overline{\mu'\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\frac{\partial u_i u_j}{\partial x_k}} + \overline{\frac{\partial\mu'}{\partial x_k}\left(u_j\frac{\partial u_k}{\partial x_i} + u_i\frac{\partial u_k}{\partial x_j}\right)}, \qquad (A.9)$$

and now for the terms depending on the fluctuating viscosity and the average rate of deformation tion

$$\overline{2u_j\frac{\partial}{\partial x_k}(\mu' S_{ik})} + \overline{2u_i\frac{\partial}{\partial x_k}(\mu' S_{jk})}$$

$$= \overline{\mu'u_j}\frac{\partial}{\partial x_k}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right) + \overline{\mu'u_i}\frac{\partial}{\partial x_k}\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right) + \overline{\frac{\partial\mu'}{\partial x_k}u_j}\frac{\partial}{\partial x_k}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)$$

$$+ \overline{\frac{\partial\mu'}{\partial x_k}u_i}\frac{\partial}{\partial x_k}\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right)$$

$$= \overline{\mu'u_j}\frac{\partial^2 U_i}{\partial x_k \partial x_k} + \overline{\mu'u_i}\frac{\partial^2 U_j}{\partial x_k \partial x_k} + \overline{u_j\frac{\partial\mu'}{\partial x_k}}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right) + \overline{u_i\frac{\partial\mu'}{\partial x_k}}\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right). \qquad (A.10)$$

Everything can now be assembled into the final form of the Reynolds stress equation

$$
\rho \frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} + \rho \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \rho \overline{u_i u_k} \frac{\partial U_j}{\partial x_k}
$$

$$
= -\rho \frac{\partial}{\partial x_k} \overline{u_i u_j u_k} - \left(\frac{\partial}{\partial x_i} \overline{p' u_j} + \frac{\partial}{\partial x_j} \overline{p' u_i} \right) + \overline{p' \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)} + \bar{\mu} \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} - 2\bar{\mu} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}
$$

$$
+ \frac{\partial \bar{\mu}}{\partial x_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \bar{\mu}}{\partial x_k} \left(\frac{\partial \overline{u_k u_j}}{\partial x_i} + \frac{\partial \overline{u_k u_i}}{\partial x_j} - 2\overline{u_k s_{ij}} \right) + \mu' \frac{\partial^2 u_i u_j}{\partial x_k \partial x_k} - 2\overline{\mu' \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} + \overline{\frac{\partial \mu'}{\partial x_k} \frac{\partial u_i u_j}{\partial x_k}}
$$

$$
+ \overline{\frac{\partial \mu'}{\partial x_k} \left(u_j \frac{\partial u_k}{\partial x_i} + u_i \frac{\partial u_k}{\partial x_j} \right)} + \overline{\mu' u_j} \frac{\partial^2 U_i}{\partial x_k \partial x_k} + \overline{\mu' u_i} \frac{\partial^2 U_j}{\partial x_k \partial x_k} + \overline{u_j \frac{\partial \mu'}{\partial x_k}} \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right)
$$

$$
+ \overline{u_i \frac{\partial \mu'}{\partial x_k}} \left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j} \right). \tag{A.11}
$$

A.3. The turbulent kinetic energy transport equation

There are two different ways to deduce this equation: either from first principles, using a methodology similar to that used here to get the Reynolds stress equation, or by contraction of indices of the Reynolds stress equation. The first strategy is adopted here. Consider again the *i*-momentum equation written in terms of the instantaneous values (Eq. (A.3)) and the time-average momentum equation for the same *i*-component (Eq. (A.2)). Eq. (A.3) is multiplied by the instantaneous velocity $(U_i + u_i)$ leading to Eq. (A.12) and (Eq. (A.2)) is multiplied by the time-average velocity U_i leading to Eq. (A.13):

$$
\rho(U_i + u_i) \left[\frac{\partial(U_i + u_i)}{\partial t} + (U_k + u_k) \frac{\partial(U_i + u_i)}{\partial x_k} \right]
$$

$$
= (U_i + u_i) \left[-\frac{\partial p}{\partial x_i} - \frac{\partial p'}{\partial x_i} \right] + (U_i + u_i) \frac{\partial}{\partial x_k} [2(\bar{\mu} + \mu')(S_{ik} + s_{ik})], \tag{A.12}
$$

$$
\rho U_i \left[\frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} \right] = -U_i \frac{\partial p}{\partial x_i} + U_i \frac{\partial}{\partial x_k} [2\bar{\mu} S_{ik} + 2\overline{\mu' s_{ik}} - \rho \overline{u_i u_k}]. \tag{A.13}
$$

Eq. (A.13) is now subtracted from Eq. (A.12) and the result is time-averaged, to yield

$$
\rho \overline{u_i \frac{\partial u_i}{\partial t}} + \rho U_k \overline{u_i \frac{\partial u_i}{\partial x_k}} + \rho \overline{u_i u_k} \frac{\partial U_i}{\partial x_k} + \rho \overline{u_i u_k \frac{\partial u_i}{\partial x_k}} = -\overline{u_i \frac{\partial p'}{\partial x_i}} + 2\overline{u_i \frac{\partial(\bar{\mu} s_{ik} + \mu' S_{ik} + \mu' s_{ik})}{\partial x_k}}. \tag{A.14}
$$

Now, a few transformations are performed: first, the turbulent kinetic energy k is defined as $k \equiv \overline{u_{ii}^2}/2$, and its fluctuation as $k' \equiv u_{ii}^2/2$. Secondly,

$$
\frac{\partial}{\partial x_k} (\overline{p' u_k}) = \overline{p' \frac{\partial u_k}{\partial x_k}} + \overline{u_k \frac{\partial p'}{\partial x_k}} = \overline{u_k \frac{\partial p'}{\partial x_k}},
$$

and given the symmetries of s_{ik} (generally $s_{ij} A_{ij} = (\partial u_i/\partial x_k) A_{ij}$) and of $\overline{u_i u_j}$, $\overline{u_i u_k}(\partial U_i/\partial x_k) = \overline{u_i u_k} S_{ik}$.

Finally, since $\partial \overline{u_i \bar{\mu} s_{ik}} / \partial x_k = \overline{u_i (\partial (\bar{\mu} s_{ik})) / \partial x_k} + \overline{\bar{\mu} s_{ik} (\partial u_i / \partial x_k)}$, back-substituting leads to the following form of the turbulent kinetic energy equation

$$\rho \frac{\mathrm{D}k}{\mathrm{D}t} = -\rho \overline{u_i u_k} S_{ik} - \frac{\overline{\partial u_i p'}}{\partial x_i} + \frac{\partial}{\partial x_k}(2\bar{\mu}\overline{u_i s_{ik}} + 2\overline{\mu' u_i} S_{ik} + 2\overline{\mu' u_i s_{ik}} - \overline{k' u_k})$$
$$- 2\bar{\mu}\overline{s_{ik} s_{ik}} - 2\overline{\mu' s_{ik} s_{ik}} - 2\overline{\mu' s_{ik}} S_{ik}. \tag{A.15}$$

A.4. The transport equation for the rate of dissipation of turbulent kinetic energy

Following Speziale [43], the transport equation for the rate of dissipation of turbulent kinetic energy is obtained as

$$\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}(Lu_i)} = 0, \tag{A.16}$$

where Lu_i is the equation of the fluctuations of the *i*-velocity component (Eq. (A.4)).

Next, applying the operators onto Eq. (A.16) and time-averaging leads to

$$\underbrace{\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t}\right)}}_{\text{I and II}} + \underbrace{\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left(\rho u_k \frac{\partial U_i}{\partial x_k}\right)}}_{\text{III}} + \underbrace{\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\rho \frac{\partial}{\partial x_k}(u_i u_k)\right]}}_{\text{IV}}$$

$$- \underbrace{\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\rho \frac{\partial}{\partial x_k}(\overline{u_i u_k})\right]}}_{\text{V}} + \underbrace{\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left(\frac{\partial p'}{\partial x_i}\right)}}_{\text{VI}} - \underbrace{\overline{4\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\bar{\mu} s_{ik})\right]}}_{\text{VII}}$$

$$- \underbrace{\overline{4\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\mu' s_{ik})\right]}}_{\text{VIII}} - \underbrace{\overline{4\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\mu' S_{ik})\right]}}_{\text{IX}} + \underbrace{\overline{4\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\overline{\mu' s_{ik}})\right]}}_{\text{X}} = 0.$$
$$\tag{A.17}$$

All terms of Eq. (A.17) contain the viscosity and hence will be different from those in the corresponding Newtonian equation. Next, each of the terms is analysed. Terms I and II can each be split into a local time variation term and an advective term. All terms will be named as either Newtonian (N) if they exist for the equivalent Newtonian equation, or as non-Newtonian (NN) if they are new.

- Term I

$$\overline{2\hat{\mu} \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m}\left(\rho \frac{\partial u_i}{\partial t}\right)} = \underbrace{\rho\bar{\mu} \frac{\partial}{\partial t}\left(\overline{\frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_m}}\right)}_{\text{Ia (N)}} + \underbrace{\rho\overline{\mu' \frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_m}\right)}}_{\text{Ib (NN)}}.$$

- Term II

$$2\rho\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(U_k\frac{\partial u_i}{\partial x_k}\right)} = 2\rho\frac{\partial U_k}{\partial x_m}\left\{\underbrace{\underbrace{\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}}}_{(N)} + \underbrace{\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}}}_{(NN)}}_{IIa}\right\}$$

$$+ \rho U_k\left\{\underbrace{\underbrace{\bar{\mu}\frac{\partial}{\partial x_k}\left[\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right]}_{(N)} + \underbrace{\overline{\mu'\frac{\partial}{\partial x_k}\left[\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]}}_{(NN)}}_{IIb}\right\}.$$

- Term III

$$2\rho\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(u_k\frac{\partial U_i}{\partial x_k}\right)} = 2\rho\frac{\partial U_i}{\partial x_k}\left[\underbrace{\underbrace{\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}}}_{(N)} + \underbrace{\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}}}_{(NN)}}_{IIIa}\right]$$

$$+ 2\rho\frac{\partial^2 U_i}{\partial x_k\partial x_m}\left[\underbrace{\underbrace{\overline{\bar{\mu}u_k\frac{\partial u_i}{\partial x_m}}}_{(N)} + \underbrace{\overline{\mu'u_k\frac{\partial u_i}{\partial x_m}}}_{(NN)}}_{IIIb}\right].$$

- Term IV

$$2\rho\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial(u_i u_k)}{\partial x_k}\right]} = \underbrace{2\rho\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}}}_{IVa\ (N)} + \underbrace{2\rho\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}}}_{IVb\ (NN)}$$

$$+ \underbrace{\rho\bar{\mu}\frac{\partial}{\partial x_k}\left(\overline{u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right)}_{IVc\ (N)} + \underbrace{\rho\overline{\mu'\frac{\partial}{\partial x_k}\left[u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]}}_{IVd\ (NN)}.$$

- Term V

$$2\rho\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \overline{u_i u_k}}{\partial x_k}\right]} = \underbrace{2\rho\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}}\frac{\partial}{\partial x_m}\left[\frac{\partial \overline{u_i u_k}}{\partial x_k}\right]}_{\text{V (NN)}}.$$

- Term VI

$$2\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial p'}{\partial x_i}\right)} = \underbrace{2\bar{\mu}\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right)}_{\text{VIa (N)}} + \underbrace{2\overline{\mu'\frac{\partial}{\partial x_i}\left(\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)}}_{\text{VIb (NN)}}.$$

- Term VII

$$4\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\mu s_{ik})\right]} = \underbrace{2\bar{\mu}\frac{\partial^2\bar{\mu}}{\partial x_m\partial x_k}\overline{\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIa (NN)}} + \underbrace{2\frac{\partial^2\bar{\mu}}{\partial x_m\partial x_k}\overline{\mu'\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIa}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}\frac{\partial\bar{\mu}}{\partial x_k}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIb (NN)}} + \underbrace{2\frac{\partial\bar{\mu}}{\partial x_k}\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIb}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}\frac{\partial\bar{\mu}}{\partial x_m}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}}_{\text{VIIc (NN)}} + \underbrace{2\frac{\partial\bar{\mu}}{\partial x_m}\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}}_{\text{VIIc}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}^2\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]}}_{\text{VIId (N)}} + \underbrace{2\bar{\mu}\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]}}_{\text{VIId}'\text{ (NN)}}.$$

- Term VIII

$$4\hat{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\mu' s_{ik})\right]} = \underbrace{2\bar{\mu}\frac{\partial^2\mu'}{\partial x_m\partial x_k}\overline{\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIIa (NN)}} + \underbrace{2\frac{\partial^2\mu'}{\partial x_m\partial x_k}\overline{\mu'\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIIa}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}\overline{\frac{\partial\mu'}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIIb (NN)}} + \underbrace{2\overline{\mu'\frac{\partial\mu'}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}}_{\text{VIIIb}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}\overline{\frac{\partial\mu'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}}_{\text{VIIIc (NN)}} + \underbrace{2\overline{\mu'\frac{\partial\mu'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}}_{\text{VIIIc}'\text{ (NN)}}$$

$$+ \underbrace{2\bar{\mu}\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]}}_{\text{VIIId (NN)}} + \underbrace{2\mu'^2\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]}}_{\text{VIIId}'\text{ (NN)}}.$$

- Term IX

$$4\hat{\mu}\,\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\mu' S_{ik})\right]} = \underbrace{2\bar{\mu}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial^2 \mu'}{\partial x_m \partial x_k}}}_{\text{IXa (NN)}}$$

$$+ \underbrace{2\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial^2 \mu'}{\partial x_m \partial x_k}}}_{\text{IXa' (NN)}} + \underbrace{2\bar{\mu}\frac{\partial}{\partial x_m}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_k}}}_{\text{IXb (NN)}}$$

$$+ \underbrace{2\frac{\partial}{\partial x_m}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_k}}}_{\text{IXb' (NN)}} + \underbrace{2\bar{\mu}\frac{\partial^2 U_i}{\partial x_k \partial x_k}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_m}}}_{\text{IXc (NN)}}$$

$$+ \underbrace{2\frac{\partial^2 U_i}{\partial x_k \partial x_k}\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_m}}}_{\text{IXc' (NN)}} + \underbrace{2\bar{\mu}\frac{\partial}{\partial x_m}\left(\frac{\partial^2 U_i}{\partial x_k \partial x_k}\right)\overline{\mu'\frac{\partial u_i}{\partial x_m}}}_{\text{IXd (NN)}}$$

$$+ \underbrace{2\frac{\partial}{\partial x_m}\left(\frac{\partial^2 U_i}{\partial x_k \partial x_k}\right)\overline{\mu'^2\frac{\partial u_i}{\partial x_m}}}_{\text{IXd' (NN)}}.$$

- Term X

$$4\hat{\mu}\,\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\overline{\mu' s_{ik}})\right]} = 4\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}(\overline{\mu' s_{ik}})\right]}$$

$$= \underbrace{2\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \mu'}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right]}}_{\text{Xa (NN)}} + \underbrace{2\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\mu'\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right]}}_{\text{Xb (NN)}}.$$

In these transformations there are several common terms, especially those associated with the Reynolds decomposition of the instantaneous viscosity.

Defining the average (ε) and fluctuating (ε') rates of dissipation as

$$\varepsilon \equiv \overline{(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}} \quad \text{and} \quad \varepsilon' \equiv (\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m},$$

the various terms can be put into a form which enables comparisons with the corresponding Newtonian equation for ε. The terms, where a transformation can be performed, are

$$\text{Terms Ia + Ib} = \overline{\rho(\bar{\mu} + \mu')\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)} = \rho\frac{\partial \varepsilon}{\partial t} - \rho\overline{\frac{\varepsilon'}{\bar{\mu} + \mu'}\frac{\partial(\bar{\mu} + \mu')}{\partial t}}.$$

$$\text{Term IIb} = \overline{\rho U_k(\bar{\mu} + \mu')\frac{\partial}{\partial x_k}\left[\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right]} = \rho U_k\frac{\partial \varepsilon}{\partial x_k} - \rho U_k\overline{\frac{\varepsilon'}{\bar{\mu} + \mu'}\frac{\partial(\bar{\mu} + \mu')}{\partial x_k}}.$$

$$\text{Terms VIId} + \text{VIIId} = 2\overline{\bar{\mu}(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]} + 2\overline{\mu'(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)\right]}$$

$$= \overline{(\bar{\mu} + \mu')^2\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)\right]} - 2\overline{(\bar{\mu} + \mu')^2\frac{\partial^2 u_i}{\partial x_k \partial x_m}\frac{\partial^2 u_i}{\partial x_k \partial x_m}}.$$

$$\text{Terms VIIa} + \text{VIIIa} = 2\overline{\frac{\partial^2(\bar{\mu} + \mu')}{\partial x_m \partial x_k}(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)}.$$

$$\text{Terms VIIb} + \text{VIIIb} = 2\overline{(\bar{\mu} + \mu')\frac{\partial(\bar{\mu} + \mu')}{\partial x_k}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)}.$$

$$\text{Terms VIIc} + \text{VIIIc} = 2\overline{(\bar{\mu} + \mu')\frac{\partial(\bar{\mu} + \mu')}{\partial x_m}\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}.$$

Finally, it is time to write the ε equation into its final form.

$$\rho\frac{\partial \varepsilon}{\partial t} - \rho\overline{\frac{\varepsilon'}{\bar{\mu} + \mu'}\frac{\partial(\bar{\mu} + \mu')}{\partial t}} + \rho U_k\frac{\partial \varepsilon}{\partial x_k} - \rho U_k\overline{\frac{\varepsilon'}{\bar{\mu} + \mu'}\frac{\partial(\bar{\mu} + \mu')}{\partial x_k}}$$

$$= -2\rho\frac{\partial U_k}{\partial x_m}\left(\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}} + \overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}}\right) - 2\rho\frac{\partial U_i}{\partial x_k}\left(\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}} + \overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_k}{\partial x_m}}\right)$$

$$- 2\rho\frac{\partial^2 U_i}{\partial x_k \partial x_m}\left(\bar{\mu}\overline{u_k\frac{\partial u_i}{\partial x_m}} + \overline{\mu'u_k\frac{\partial u_i}{\partial x_m}}\right) - 2\rho\bar{\mu}\overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}} - 2\rho\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_m}}$$

$$- \rho\bar{\mu}\frac{\partial}{\partial x_k}\left(\overline{u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right) - \rho\overline{\mu'\frac{\partial}{\partial x_k}\left(u_k\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)} + 2\rho\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial \overline{u_i u_k}}{\partial x_k}\right)}$$

$$- 2\bar{\mu}\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}}\right) - 2\overline{\mu'\frac{\partial}{\partial x_i}\left(\frac{\partial p'}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)} + 2\overline{\frac{\partial^2(\bar{\mu} + \mu')}{\partial x_m \partial x_k}(\bar{\mu} + \mu')\left[\frac{\partial u_i}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right]}$$

$$+ 2\overline{\frac{\partial(\bar{\mu} + \mu')}{\partial x_k}(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} + 2\overline{\frac{\partial(\bar{\mu} + \mu')}{\partial x_m}(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}\right)}$$

$$+ \overline{(\bar{\mu} + \mu')^2\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_m}\right)\right]} - 2\overline{(\bar{\mu} + \mu')^2\frac{\partial^2 u_i}{\partial x_k \partial x_m}\frac{\partial^2 u_i}{\partial x_k \partial x_m}}$$

$$+ 2\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial^2 \mu'}{\partial x_m \partial x_k}} + 2\frac{\partial}{\partial x_m}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right)\overline{(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_k}}$$

$$+ 2\frac{\partial^2 U_i}{\partial x_k \partial x_k}\overline{(\bar{\mu} + \mu')\frac{\partial u_i}{\partial x_m}\frac{\partial \mu'}{\partial x_m}} + 2\frac{\partial}{\partial x_m}\left(\frac{\partial^2 U_i}{\partial x_k \partial x_k}\right)\overline{(\bar{\mu} + \mu')\mu'\frac{\partial u_i}{\partial x_m}}$$

$$- 2\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\frac{\partial \mu'}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right]} - 2\overline{\mu'\frac{\partial u_i}{\partial x_m}\frac{\partial}{\partial x_m}\left[\mu'\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right]}. \tag{A.18}$$

In conclusion, the transport equations are the following: for momentum, Eq. (A.2); for the Reynolds stresses, Eq. (A.11); for the turbulent kinetic energy, Eq. (A.15) and for the rate of dissipation of turbulent kinetic energy, Eq. (A.18).

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