RECENT DEVELOPMENTS AND CHALLENGES IN TURBULENCE MODELING FOR VISCOELASTIC FLUIDS

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• Brief review of existing RANS models for FENE-P

• Governing equations for FENE-P in RANS/RACE form
  (Reynolds decomposition)

• Development of closures for RANS/RACE of FENE-P
  (2007 (LDR) and 2008 (LDR & HDR) closures)

• Some results (2007 models only)

• Conclusions and future prospects
Governing equations: turbulent flow & FENE-P

**Reynolds decomposition**

\[ \hat{B} = B + b' \quad \text{where} \quad \overline{b'} = 0 \]

- Instantaneous quantities
- Overbar or upper-case letters - time-averaged quantities
- Lower-case letters - fluctuating quantities

**Continuity (incompressible):**

\[ \frac{\partial \hat{U}_i}{\partial x_i} = 0 \]

**Momentum** \( \mathcal{M}(\hat{U}_{ij}) \):

\[ \rho \frac{\partial \hat{U}_i}{\partial t} + \rho \hat{U}_k \frac{\partial \hat{U}_i}{\partial x_k} = - \frac{\partial \hat{p}}{\partial x_i} + \eta_s \frac{\partial^2 \hat{U}_i}{\partial x_k \partial x_k} + \frac{\partial \hat{\tau}_{ik,p}}{\partial x_k} \]

**Rheological constitutive equation:** FENE-P

\[ \hat{\tau}_{ij} = 2\eta_s \hat{S}_{ij} + \hat{\tau}_{ij,p} \]

\[ f\left(\hat{C}_{kk}\right) = \frac{L^2 - 3}{L^2 - \hat{C}_{kk}} \quad f\left(L\right) = 1 \]

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**Solution of governing equations**

**Direct Numerical Simul.**

- Sureshkumar, Beris, Handler (1997) PoF, v9, 743
- Den Toonder, Hulsen, Kuiken, Nieuwstadt (1997) JFM 337, 193
- Dimitropoulos, Sureshkumar, Beris (1998) JNNFM v79, 433
- Dimitropoulos, Sureshkumar, Beris, Handler (2001) PoF v13, 1016
- Ptasinski et al (2003) JFM v490, 251
- Yu, Kawaguchi (2003) IJHFF v24, 491
- Yu, Li, Kawaguchi (2004) IJHFF v25, 961
- Dimitropoulos et al (2005) PoF v17, 011705
- Li, Gupta, Sureshkumar, Khomami (2006) JNNFM v139, 177
- Li, Sureshkumar, Khomami (2006) JNNFM v140, 23
- Benzi, Angelis, L’vov, Procaccia, Tiberkevich (2006) JFM v551, 185
- & others — see recent review

**Physical understanding**

**Too costly for engineering calculations**

**Turbulence model development**

- LES
- RANS/RACE

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Time-averaged governing equations: RANS and RACE

Continuity: \[
\frac{\partial U_i}{\partial x_i} = 0
\]

Momentum balance:
\[
\rho \frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \eta_s \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} \left( \rho u_i u_k \right) + \frac{\partial \tau_{ik,p}}{\partial x_k}
\]

Rheological constitutive equation: FENE-P
\[
\bar{\tau}_{ij,p} = \eta_p \left[ f(C_{kk}) C_{ij} - f(L) \delta_{ij} \right] + \frac{\eta_p}{\lambda} f(C_{kk} + c_{kk}) c_{ij}
\]

RACE
\[
\nabla \cdot C_{ij} + u_k \frac{\partial c_{ij}}{\partial x_k} - \left( c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k} \right) = - \frac{\bar{\tau}_{ij,p}}{\eta_p}
\]

Closures required
Existing models for FENE-P: Li et al (2006)

\[
\rho \frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \eta_s \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k} \left( \rho u_i u_k \right) + \frac{\partial \tau_{ik,p}}{\partial x_k}
\]

**Reynolds stress**

\[
-uv = v_{T,v} \frac{dU}{dy}
\]

\[
v_{T,v} = \phi v_{T,N}
\]

\[
v_{T,N} = \kappa u_\tau y
\]

\[
\phi = \left[ a(DR) y + b(DR) \right]
\]

**0 equation model (shear stress only)**

\[
\int_0^{R_e} \tau_{xy,p} dy = \int_0^{R_e} M_{xy} dy + \int_0^{R_e} NLT_{xy} dy
\]

\[
I_{M_{xy}} = a' + b' DR + c' DR^2
\]

\[
I_{NLT_{xy}} = a'' + b'' DR + c'' DR^2
\]

\[
DR = 80 \left\{ 1 - \exp \left[ -0.025(We_\tau - 6.25) \left( \frac{Re_\tau}{125} \right)^{-0.225} \right] \right\} \left[ 1 - \exp(-0.0275L) \right]
\]
Existing models for FENE-P

**FENE-P and based on DNS**

Leighton, Walker and Stephens (2002) APS meeting?

- Reynolds stress transport model
- Slow pressure-strain redistribution term is modified by polymer (limits energy redistribution)
- New term in RS equation: interaction of $\tau'_{p,ij}$ & turbulence
- New term in $C_{ij}$ equation ($NLT_{ij}$)
- Additional diffusive flux terms not modeled

**Shaqfeh (2006) AIChE Conference**

- $k-\varepsilon v^2-f$ extension model of Durbin (1995)
- Simplified model: $\bar{\tau}_{p,ij}$ proportional to mean strain (elongation)
- Coefficient has laminar and turbulent contribution
- Laminar part proportional to $\partial U/\partial y$
- Turbulent part proportional to $k$
- Modifies pressure strain ($v^2$ equation)
- One transport equation for $C_{kk}$
DNS cases: channel flow

Fully-developed channel flow

\[ u \]

\[ 2, y \]

\[ 1, x \]

\[ 2h \]

\[ We_\tau = \frac{\lambda u_\tau^2}{v_0} \]

\[ Re_\tau = \frac{hu_\tau}{v_0} \]

\[ Re_\tau = 395, \beta = 0.9, L^2 = 900 \]

**Low Drag Reduction**

\[ We_\tau = 25, DR = 18\% \]

**High Drag Reduction**

\[ We_\tau = 100, DR = 37\% \]

- 2007 models (Pinho et al JNNFM 2008 & unpublished) - Only LDR
- 2008 model (under develop.)- Recalculated DNS + LDR & HDR
- Closures valid for 1\textsuperscript{st} & higher order turbulence models
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Function $f(C_{kk})$

Function: $f(C_{kk}) = \frac{L^2 - 3}{L^2 - (C_{kk} + c'_{kk})}$

$C_{kk} > \sqrt{c'_{kk}^2}$

$f(C_{kk})b_{lm}d_i \approx f(C_{kk})b_{lm}d_i$

$f(C_{kk})b_{ij} \approx f(C_{kk})\bar{b}_{ij} = 0$

This will be used frequently
Time-average evolution equation for the conformation: RACE

\[
\lambda \nabla C_{ij} + \lambda \left[ u_k \frac{\partial c_{ij}}{\partial x_k} - \left( c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k} \right) \right] = - \left[ f(C_{kk}) C_{ij} - f(L) \delta_{ij} \right]
\]

\[
\nabla C_{ij} + u_k \frac{\partial c_{ij}}{\partial x_k} - \left( c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k} \right) + D_{ij} = - \frac{\tau_{ij,p}}{\eta_p}
\]


- Oldroyd derivative
- Mean flow distortion
- Exact and large

:turbulent distortion:
- originates in distortion of Oldroyd derivative- not negligible
- Must be modeled

- originates in advective term, negligible
- no need for modeling

\[ D_{ij} = \frac{\tau_{ij,p}}{\eta_p} \]

\[ \tau_{ij,p} \]

Added for stability
Should be negligible
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Approximate equation for $NLT_{ij}$

$$NLT_{ij} = c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k}$$

$$\mathcal{L}(\hat{C}_{kj}) \frac{\partial u_i}{\partial x_k} + \mathcal{L}(\hat{C}_{ik}) \frac{\partial u_j}{\partial x_k}$$

SIMPLER THAN EXACT EQUATION:

$$f(\hat{C}_{mm}) c_{kj} \frac{\partial u_i}{\partial x_k} + f(\hat{C}_{mm}) c_{ik} \frac{\partial u_j}{\partial x_k} + C_{kj} f(\hat{C}_{mm}) \frac{\partial u_i}{\partial x_k} + C_{ik} f(\hat{C}_{mm}) \frac{\partial u_j}{\partial x_k} + \frac{\partial C_{kj}}{\partial x_k} \frac{\partial u_i}{\partial x_k} + \frac{\partial C_{ik}}{\partial x_k} \frac{\partial u_j}{\partial x_k} + u_{i} \frac{\partial C_{kj}}{\partial x_k} + u_{n} \frac{\partial C_{ik}}{\partial x_k}$$

$$- \lambda \left[ \frac{\partial U_{i}}{\partial x_{n}} \left( c_{j} \frac{\partial u_{i}}{\partial x_{k}} + c_{i} \frac{\partial u_{j}}{\partial x_{k}} \right) + \frac{\partial U_{i}}{\partial x_{n}} \frac{\partial u_{i}}{\partial x_{k}} + \frac{\partial U_{i}}{\partial x_{n}} \frac{\partial u_{j}}{\partial x_{k}} + C_{i} \frac{\partial u_{i}}{\partial x_{k}} \frac{\partial u_{j}}{\partial x_{k}} \right]$$

$$- \lambda \left[ C_{j} \frac{\partial u_{i}}{\partial x_{n}} \frac{\partial u_{i}}{\partial x_{k}} + C_{i} \frac{\partial u_{i}}{\partial x_{n}} \frac{\partial u_{j}}{\partial x_{k}} + \frac{\partial u_{i}}{\partial x_{n}} \frac{\partial u_{i}}{\partial x_{k}} + \frac{\partial u_{j}}{\partial x_{n}} \frac{\partial u_{j}}{\partial x_{k}} + C_{i} \frac{\partial u_{i}}{\partial x_{k}} \frac{\partial u_{j}}{\partial x_{k}} \right] = 0$$

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Simplifications for modeling $NLT_{ij}$

$$f' c'_{jk} \frac{\partial u_i}{\partial x_k} + f' c'_{ik} \frac{\partial u_j}{\partial x_k} \approx f(C_{mm}) \left( c_{kj} \frac{\partial u_i}{\partial x_k} + c_{ik} \frac{\partial u_j}{\partial x_k} \right) = f(C_{mm})^{NLT}_{ij}$$
**Previous models for $NLT_{ij}$ - 2007 1st model**

**First model**

Exact equation is too complex.

Alternative model based on:

1) Identification of possible dependencies from inspection of exact equation

2) Simplicity, but capturing main features

3) $We=25$ (DR=18%)

$$f\left(C_{mm}\right)^{NLT}_{ij} = \frac{\lambda}{\mu} \text{function}\left(S_{ij}, W_{ij}, C_{ij}, \varepsilon^N_{ij}, \frac{\partial u_i u_j}{\partial x_k}, \frac{\partial C_{ij}}{\partial x_k}, \frac{\partial NLT_{ij}}{\partial x_n}, M_{ij}, u_i u_j\right)$$

$$f\left(C_{mm}\right)^{NLT}_{ij} = f_{\mu_1} \left[ \frac{C_{E_3}}{v_0^2} \frac{u_i u_j^2}{C_{kk}} + \frac{C_{\alpha_{14}}}{v_0} (u_i W_{kn} C_{nj} + u_j W_{kn} C_{ni} + u_k W_{jn} C_{nk}) \right]$$

2 coefficients

$C_{E_3} = 0.00035; C_{\alpha_{14}} = 0.00015$

1 damping function

$$f_{\mu_1} = \left(1 - \exp\left(-y^+ / 26.5\right)\right)^2$$

Pinho, Li, Younis, Sureshkumar (2008) JNNFM, in press
Modeling $NLT_{ij}$ 1- 2008 model and 2$^{nd}$ 2007 model

1) $CT_{ij} = u_k \frac{\partial c_{ij}}{\partial x_k} \approx 0$ suggests $u_n \frac{\partial c_{ij}}{\partial x_k} \approx 0$

2) Homogeneous turbulence (negligible turbulent diffusion): $u_n \frac{\partial u_j}{\partial x_k} \approx 0$

3) Invariance laws $U_n \left( \frac{\partial c_{kj}}{\partial x_n} \frac{\partial u_i}{\partial x_k} + \frac{\partial c_{ik}}{\partial x_n} \frac{\partial u_j}{\partial x_k} \right) = 0$

4) Homogeneous isotropic turbulence

$\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} = \frac{8}{3} k \left[ \delta_{ij} \delta_{kl} - \frac{1}{4} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right]$

Taylor’s longitudinal micro-scale $\varepsilon = 20 \frac{\nu k}{\lambda_f^2}$

$C_{mn} \left( \frac{\partial u_j}{\partial x_n} \frac{\partial u_i}{\partial x_k} + \frac{\partial u_i}{\partial x_n} \frac{\partial u_j}{\partial x_k} \right) + C_{jn} \frac{\partial u_k}{\partial x_n} \frac{\partial u_i}{\partial x_k} + C_{in} \frac{\partial u_k}{\partial x_n} \frac{\partial u_j}{\partial x_k} \approx C \varepsilon_n \int_{N_2} \frac{4}{15} \frac{\varepsilon}{\beta \times W e_{\tau_0} \times \nu_T} C_{mn} \delta_{ij}$

Hinze (1975); Mathieu & Scott (2000)
5) *ad-hoc* + symmetry, invariance, permutation, realizability

\[
\frac{\partial U_k}{\partial x_n} \left( c_{jn} \frac{\partial u_i}{\partial x_k} + c_{in} \frac{\partial u_j}{\partial x_k} \right) + \frac{\partial U_j}{\partial x_n} c_{kn} \frac{\partial u_i}{\partial x_k} + \frac{\partial U_i}{\partial x_n} c_{kn} \frac{\partial u_j}{\partial x_k} \approx \left[ \frac{\partial U_j}{\partial x_k} \frac{\partial U_m}{\partial x_n} \right] c_{N_3} \frac{u_i u_m}{\nu_0 \sqrt{2 S_{pq} S_{pq}}} + \left[ \frac{\partial U_i}{\partial x_k} \frac{\partial U_m}{\partial x_n} \right] c_{kn} \frac{u_j u_m}{\nu_0 \sqrt{2 S_{pq} S_{pq}}}
\]

\[
C_{kj} f(\hat{C}_{mm}) \frac{\partial u_i}{\partial x_k} + C_{ik} f(\hat{C}_{mm}) \frac{\partial u_j}{\partial x_k} \approx C_{kj} f(C_{mm}) \frac{\partial u_i}{\partial x_k} + C_{ik} f(C_{mm}) \frac{\partial u_j}{\partial x_k} = 0
\]

\[
C_{N_2} \left[ C_{kj} f(C_{mm}) \frac{\partial U_i}{\partial x_k} + C_{ik} f(C_{mm}) \frac{\partial U_j}{\partial x_k} \right]
\]

6) Decoupling 3rd order correlation

\[
c_{jn} \frac{\partial u_k}{\partial x_n} \frac{\partial u_i}{\partial x_k} + c_{in} \frac{\partial u_k}{\partial x_n} \frac{\partial u_j}{\partial x_k} + c_{kn} \frac{\partial u_j}{\partial x_n} \frac{\partial u_i}{\partial x_k} + c_{kn} \frac{\partial u_i}{\partial x_n} \frac{\partial u_j}{\partial x_k} \approx -C_{N_4} f_{N_1} \left[ C_{jn} \frac{\partial U_k}{\partial x_n} \frac{\partial U_i}{\partial x_k} + C_{in} \frac{\partial U_k}{\partial x_n} \frac{\partial U_j}{\partial x_k} + C_{kn} \frac{\partial U_j}{\partial x_n} \frac{\partial U_i}{\partial x_k} + C_{kn} \frac{\partial U_i}{\partial x_n} \frac{\partial U_j}{\partial x_k} \right]
\]
Closure for $NLT_{ij}$-2008 model

$\text{We}=25$ (LDR) & $100$ (HDR)

Modified term & reprocessed DNS data

\[
f(C_{mm}) \frac{NLT_{ij}}{\lambda} = f(C_{mm}) \frac{C_{ij}f(C_{mm})}{\lambda \text{We}_{\tau_0}} - C_{N_2} \left[ C_{kj} \frac{\partial U_i}{\partial x_k} + C_{ik} \frac{\partial U_j}{\partial x_k} \right] \\
+ C_{N_3} \frac{C_{kn}}{\nu_0 \sqrt{2S_{pq} S_{pq}}} \left[ \frac{\partial U_j}{\partial x_m} \frac{\partial U_m}{\partial x_n} + \frac{\partial U_i}{\partial x_m} \frac{\partial U_m}{\partial x_n} \right] \\
- C_{N_4} f_{N_1} \left[ C_{jn} \frac{\partial U_k}{\partial x_n} \frac{\partial U_i}{\partial x_k} + C_{in} \frac{\partial U_k}{\partial x_n} \frac{\partial U_j}{\partial x_k} + C_{kn} \left( \frac{\partial U_j}{\partial x_n} \frac{\partial U_i}{\partial x_k} + \frac{\partial U_i}{\partial x_n} \frac{\partial U_j}{\partial x_k} \right) \right] \\
+ C_{N_5} f_{N_2} \frac{4}{15} \frac{\varepsilon}{\beta v_s \text{We}_{\tau_0}} C_{mm} \delta_{ij}
\]

\[
f_{N_1} = \left[ 1 - 0.8 \exp(-y^+/30) \right]^2 \\
f_{N_2} = \left[ 1 - \exp(-y^+/25) \right]^4
\]

$C_{N_1} = 12.7$ \quad $C_{N_4} = 1.11$  
$C_{N_2} = 0.32$ \quad $C_{N_5} = 1.13$  
$C_{N_3} = 0.024$

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2007 models for $NLT_{ij}$: First versus second model

$We=25$, $DR=18\%$
Performance of the 2008 $NLT_{ij}$ closure: $xx$ and $yy$

**xx**

$NLT_{xx}$
- DNS
- Model
- DNS
- Model

- $We=25$, $DR=18$
- $We=100$, $DR=37$

**yy**

$NLT_{yy}$
- DNS
- Model
- DNS
- Model

- $We=25$, $DR=18$
- $We=100$, $DR=37$

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Performance of the 2008 $NLT_{ij}$ closure: trace and xy

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Modeling the Reynolds stress

**Major issue: what model to use for Reynolds stresses?**

1) Reynolds stresses: Prandtl-Kolmogorov model (k-ε closure)

\[-u_iu_j = 2\nu T S_{ij} - \frac{2}{3}k\delta_{ij}\]

with \[\nu T = C_\mu f_\mu \frac{k^2}{\varepsilon^N + \varepsilon^V}\]

2) Dissipation of turbulent kinetic energy: \(\varepsilon^N\)

\[
2\nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \rho \frac{Du_i}{Dt} \right) + 2\nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \rho u_i \frac{\partial U}{\partial x_k} \right) + 2\nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \rho \frac{\partial u_i u_k}{\partial x_k} \right) + 2\nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \frac{\partial p'}{\partial x_i} \right) - 2\rho \nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \frac{\partial^2 u_i}{\partial x_i \partial x_k} \right) - 2\nu \frac{\partial u_i}{\partial x_m} \frac{\partial}{\partial x_m} \left( \frac{\partial \tau'_{ik,p}}{\partial x_k} \right) = 0
\]

*New term (will be considered in the future)*

As for Newtonian fluids, most terms in \(\varepsilon^N\) are approximated
Transport equation for turbulent kinetic energy

\[
\frac{Dk}{Dt} = -\rho u_i u_k \frac{\partial U_i}{\partial x_k} - \rho u_i \frac{\partial k}{\partial x_i} - \frac{\partial p'u_i}{\partial x_i} + \eta_s \frac{\partial^2 k}{\partial x_i \partial x_i} - \eta_s \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} + \frac{\partial \tau_{ik,p}}{\partial x_k} \frac{\partial u_i}{\partial x_k} - \tau_{ik,p} \frac{\partial u_i}{\partial x_k}
\]

When \( We \) increases (\( DR \uparrow \))

- \( P_k \) decreases
- \( \varepsilon^N \) decreases
- \( Q^V \) increases in buffer \( l. \), but remains small
- \( \varepsilon^V \) increases in inertial \( l. \)

Need to model well \( \varepsilon^V \)

\( Q^V \) is small

Need to modify model of \( \varepsilon^N \)
Zoom of balance of $k$: inertial sub-layer
Assumptions for viscoelastic stress work: $\varepsilon^V$

$$\varepsilon^V \equiv \frac{1}{\rho} \tau_{ik,p} \frac{\partial u_i}{\partial x_k} = \eta_p \left( C_{ik} f \left( C_{mm} + c_{mm} \right) \frac{\partial u_i}{\partial x_k} + c_{ik} f \left( C_{mm} + c_{mm} \right) \frac{\partial u_i}{\partial x_k} \right)$$

$C_{ik} f \left( C_{mm} + c_{mm} \right) \frac{\partial u_i}{\partial x_k} \ll c_{ik} f \left( C_{mm} + c_{mm} \right) \frac{\partial u_i}{\partial x_k}$

Except in viscous sublayer and buffer, but here $\varepsilon^V$ is not important
Further assumptions for viscoelastic stress work: $\varepsilon^V$

\[
f'c'_{ik} \frac{\partial u_i}{\partial x_k} \approx C_{\varepsilon^V} \times f(C_{mm}) c_{ik} \frac{\partial u_i}{\partial x_k}
\]

$C_{\varepsilon^V} \approx O(1)$

at $We_{\tau_0} = 25$

but larger as DR increases

\[\text{This is } NLT_{ii}\]
Viscoelastic stress work model

\[ \varepsilon^V \approx \frac{\eta_p}{\rho \lambda} C_{\varepsilon^V} f(C_{mm}) c_{ik} \frac{\partial u_i}{\partial x_k} = C_{\varepsilon^V} \left( \frac{W e_{\tau_0}}{25} \right)^n \frac{\eta_p}{\rho \lambda} f(C_{mm}) \left( \frac{N L T_{ii}}{2} \right) \]

\[ W e_{\tau_0} = 25 \rightarrow 1.27 \]
\[ W e_{\tau_0} = 100 \rightarrow 1.56 \]

\[ C_{\varepsilon^V} = 1.27 \]
\[ n = 1.15 \]

\[ \varepsilon^V \left( R e_{\tau_0} \right)^2 \]
versus
\[ C_{\varepsilon^V} \frac{R e_{\tau_0} (1 - \beta)}{W e_{\tau_0}} f(C_{ii}) N L T^*_{jj} \]

Previous model: \( C_{\varepsilon^V} = 1.076 \)  
(\( W e_{\tau_0} = 25 \) only)  

Pinho, Li, Younis, Sureshkumar (2008) JNNFM, in press
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**Viscoelastic turbulent transport: \( Q^v \)**

\[
Q^v \equiv \frac{\partial \tau_{ik,p}^v}{\partial x_k} = \frac{\eta_p}{\lambda} \frac{\partial}{\partial x_k} \left[ C_{ik} f(C_{mm} + c_{mm}) u_i \right] + \frac{c_{ik} f(C_{mm} + c_{mm}) u_i}{\lambda}
\]

\[
\frac{f(C_{mm}) C_{U_{ik}}}{\lambda} = f_{\mu_2} \left( \frac{25}{W_{e_0}} \right)^{0.53} \left[ -C_{\beta_1} \left( u_i u_m \frac{\partial C_{kj}}{\partial x_m} + u_j u_m \frac{\partial C_{ik}}{\partial x_m} \right) - \frac{C_{\beta_1}}{\lambda} \right] f(C_{mm}) \left[ \pm \sqrt{u_j^2 C_{ik} \pm u_i^2 C_{jk}} \right]
\]

\[
f_{\mu_2} = 1 - \exp(-y^+ / 26.5)
\]

\[
C_{\beta_1} = 1.1; C_{\beta_7} = 0.3
\]

\[
C_{FU_{ik}} = C_{ik} f(C_{mm} + c_{mm}) u_i
\]

\[= \frac{C_{FU}}{2} \frac{\lambda}{W_{e_0}} f(C_{mm}) C_{kn} \frac{\partial u_i u_i}{\partial x_n}
\]

Closure development followed similar procedures as that for \( NLT_{ij} \)
Final equations for channel flow: RANS and RACE

Momentum:
\[
\frac{d}{dy} \left[ \eta_s \frac{dU}{dy} + \overline{\tau}_{p,xy} - \rho uv \right] - \frac{dp}{dx} = 0
\]

\[
\overline{\tau}_{xy,p} = \frac{\eta_p}{\lambda} f \left( C_{kk} \right) C_{xy}
\]

\[
f \left( C_{kk} \right) C_{xy} = \lambda C_{yy} \frac{dU}{dy} + \lambda NLT_{xy}
\]

\[
f \left( C_{kk} \right) C_{yy} = \lambda NLT_{yy} + 1
\]

\[
f \left( C_{kk} \right) C_{xx} = 2 \lambda C_{xy} \frac{dU}{dy} + \lambda NLT_{xx} + 1
\]

\[
f \left( C_{kk} \right) C_{zz} = \lambda NLT_{zz} + 1
\]

Reynolds stress:
\[-\rho uv = \rho \nu_T \frac{dU}{dy} \quad \text{with} \quad \nu_T = C_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}^N + \varepsilon^V} \quad \text{and} \quad f_\mu = \left[ 1 - \exp \left( \frac{-y^+}{26.5} \right) \right]^2\]
Recent developments and challenges in turbulence modeling of viscoelastic fluids

Resende, Pinho, Kim, Younis & Sureshkumar

CEFT-FEUP Centro de Estudos de Fenómenos de Transporte, Universidade do Porto

Complex Flows of Complex Fluids, Liverpool, UK

$k$ and $\varepsilon$ transport equations: modified Nagano & Hishida

Based on Newtonian model of Nagano & Hishida (1984)

\[
0 = \frac{d}{dy} \left[ \left( \eta_s + \frac{\rho f_T v_T}{\sigma_k} \right) \frac{dk}{dy} \right] + P_k - \rho \tilde{\varepsilon}^N - \rho D^N + \eta_p \frac{d}{dy} \left[ \frac{f(C_{mm}) C u_{nm}}{\lambda} \right] - \eta_p \frac{f(C_{mm}) NLT_{nn}}{\lambda} \frac{d\sqrt{k}}{dy}^2
\]

\[
\sigma_k = 1.1 \\
\varepsilon^N = \tilde{\varepsilon}^N + D^N \\
D^N = 2\eta_s \left( \frac{d\sqrt{k}}{dy} \right)^2
\]


\[
0 = \frac{d}{dy} \left[ \left( \eta_s + \frac{\rho f_T v_T}{\sigma_\varepsilon} \right) \frac{d\tilde{\varepsilon}^N}{dy} \right] + \rho f_1 C_{\varepsilon_1} \frac{\tilde{\varepsilon}^N P_k}{\rho k} - \rho f_2 C_{\varepsilon_2} \frac{\varepsilon^{N_2}}{k} + \rho E + E_{\tau_p}
\]

\[
\sigma_\varepsilon = 1.3 \\
f_1 = 1 \\
f_2 = 1 - 0.3 \exp \left( -R_T^2 \right)
\]

\[
C_{\varepsilon_1} = 1.45 \\
C_{\varepsilon_2} = 1.90 \\
E = \frac{\eta_s}{\rho} v_T \left( 1 - f_\mu \right) \left( \frac{d^2 U}{dy^2} \right)^2
\]

\[
E_{\tau_p} = 0
\]
Predictions $U^+: Re_{\tau_0} = 395; We_{\tau_0} = 25; \beta=0.9, L^2=900$
Predictions $NLT_{ii}$: $Re_\tau = 395; \ We_\tau = 25; \ \beta = 0.9, L^2 = 900$

Models fitted to DNS

Code diverges

Models modified with simulations

Model 2

Coefficients $C_{N_1}$ & $C_{N_2}$ reduced by 4

Coefficient $C_{N_3}$ increased 60%

Coefficient $C_{N_5}$ reduced 30%
Predictions $k$ & $\varepsilon^N$: $Re_\tau = 395$; $We_\tau = 25$; $\beta = 0.9$, $L^2 = 900$
Recent developments and challenges in turbulence modeling of viscoelastic fluids

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Predictions $u'v'$ & $\tau_{p,xy}$: $Re_{\tau_0} = 395$; $We_{\tau_0} = 25$; $\beta=0.9$, $L^2=900$
Conclusions and Acknowledgments

- Closures for Low DR and High DR
- Closures for $NLT_{ij}$, $\varepsilon^V$ and $Q^V$ (in fact for $\varepsilon_{ij}^V$ and $Q_{ij}^V$)
- Developed simple low Reynolds $k$-$\varepsilon$ model works reasonably well
- Need to incorporate with better Reynolds stress closures:
  - $k$-$\omega$, modified $k$-$\varepsilon$ or $k$-$\omega$, Menter’s SST or Durbin’s $v2$-$f$
  - or RS transport (deficiencies in base model are imp.)
- Need to extend models to Maximum DR, $\beta$ & $L^2$
- DNS in other canonical flows required for extension of turb. models

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