Nondegenerate and Normal forms of the Maximum Principle for Control Problems with State Constraints

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Outline

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Optimal Control in Critical Decision Making Scenarios

- In the early years, optimal control was mainly used for planning, solved off-line (e.g. devising spaceship trajectories, economic growth, fishing policies, ...)
- Recently, it is increasingly being used in real-time to control processes (mainly within MPC, e.g. control of destillation columns in the petrol refining industry, ...) .
- More recently, its is being used in real-time to make decisions autonomously (e.g. autonomous underwater vehicles: minimum energy consumption is a key factor in data-gathering missions, communication is limited; optimal storage strategies in electrical power systems: requires fast, autonomous decisions for integration of variable and uncertain renewable generation

$\Rightarrow$ When NCO are being used, we must guarantee that they help selecting the minimizers

Necessary Conditions of Optimality (NCO), Degeneracy and Abnormality

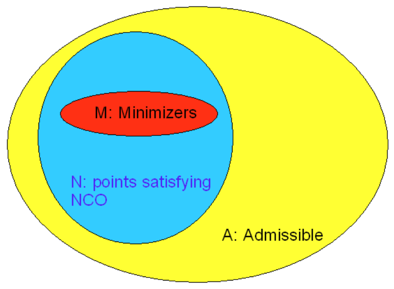
Structure: If $\bar{x}$ is a minimizer, then it satisfies the NCO.

(Example in unconstrained function optimization:
If $\bar{x}$ is a minimizer for f, then $\nabla f(\bar{x}) = 0$)

Aim: To identify a ("small") set containing all the minimizers.
(The stricter the better. Ideally, if the NCO are also sufficient, then $M = N$.)

Degeneracy phenomenon: All admissible solutions satisfy the NCO:
$N = A$.
(NCO are useless in this case.)

Abnormality phenomenon: the NCO are merely state a relation between the constraints and do not use the objective function to select candidates to minimizers.



Outline

Necessary Conditions of Optimality in Mathematical Programming

Nonlinear Programming problem with inequality constraints:

$$\begin{array}{lll} \text{(MP) Minimize}_{x\in\mathbb{R}^n} & g(x) & \\ \text{subject to} & h_i(x) \leq 0 & i = 1, 2, \ldots, k\,. \end{array}$$

Fritz-John Necessary Conditions of Optimality (1948): If $\bar{x}$ solves (MP) then $\exists(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^k$ s.t.

$$\begin{aligned} &(\lambda, \mu) \neq 0 \\ &\lambda, \mu_i \geq 0 \quad i = 1, 2, \ldots, k \\ &\lambda \nabla g(\bar{x}) + \sum_{i=1,2,\ldots,k} \mu_i \nabla h_i(\bar{x}) = 0 \\ &\sum_{i=1,2,\ldots,k} \mu_i h_i(\bar{x}) = 0\,. \end{aligned}$$

Is this useful if $\lambda = 0$? Can we choose $\lambda > 0$? When?

Can we always choose λ positive ?

Ans: Not always

A counter-example (Kuhn-Tucker 1951)

$$(MP1) \quad \text{Minimize} \quad -x_1$$

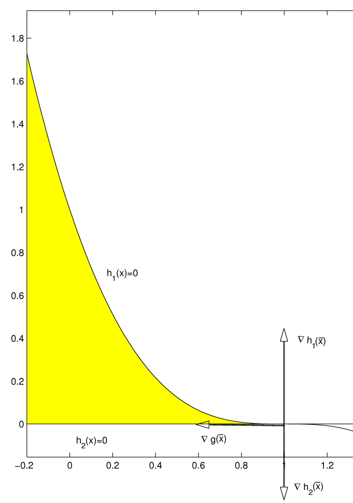
subject to

$$x_2 + (x_1 - 1)^3 \leq 0$$
$$-x_2 \leq 0.$$

Solution is $\bar{x} = (x_1, x_2) = (1, 0)$.

With $\lambda > 0$ it is impossible to satisfy

$$\lambda \nabla g(\bar{x}) + \mu_1 \nabla h_1(\bar{x}) + \mu_2 \nabla h_2(\bar{x}) = 0.$$



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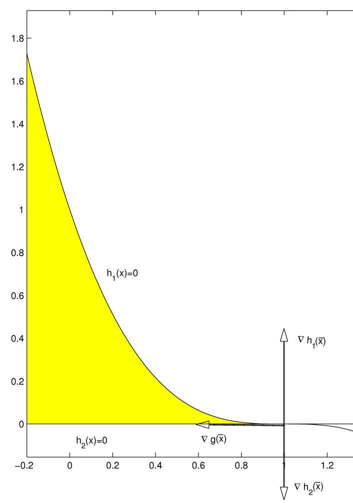


We can choose $\lambda > 0$, ($\lambda = 1$)
for all problems satisfying a
Constraint Qualification
[Kuhn Tucker 1951] (one of the most cited results in optimization)

For example

Constraint Qualification (Mangasarian-Fromovitz) There is a vector $v \in \mathbb{R}^k$ satisfying

$$\nabla h_i(\bar{x}) \cdot v < 0$$

for all i such that $h_i(\bar{x}) = 0$.

Outline

The Quest

In Optimal Control, the NCO - the Pontryangin Maximum Principle (PMP) - can also be abnormal or degenerate.

Our quest is to devise Nondegenerate and Normal forms of the PMP, valid under a constraint qualifications;

Obtain for Optimal Control a result analogous to what Kunh and Tucker have done for Nonlinear Programming.

Outline

The problem

Optimal control problem with (pathwise inequality) state constraints:

$$
\begin{aligned}
&\text{Minimize} && g(x(1)) \\
&\text{subject to} && \\
& && \dot{x}(t) = f(t, x(t), u(t)) \\
& && x(0) = x_0 \\
& && x(1) \in C \\
& && u(t) \in \Omega(t) \\
& && h(t, x(t)) \leq 0
\end{aligned}
$$

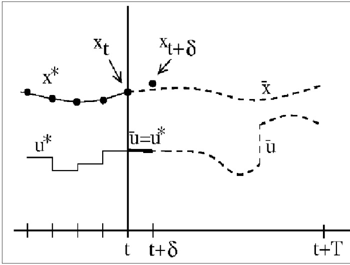
Model Predictive Control

Consider a sequence $\{t_i\}_{i\geq 0}$ s.t. $t_{i+1} = t_i + \delta$, $\delta > 0$:

1. Measure state of the plant x_{t_i}
2. Get $\bar{u} : [t_i, t_i + T] \mapsto \mathbb{R}^m$ solution to the OCP:

$$\begin{array}{lll} \text{Minimize} & \displaystyle\int_{t_i}^{t_i+T} L(t, x(t), u(t))dt + W(x(t_i + T)) & \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_i, t_i + T] \\ & x(t_i) = x_{t_i} & \\ & u(t) \in U(t) & \text{a.e. } t \in [t_i, t_i + T] \\ & x(t) \in X & \text{a.e. } t \in [t_i, t_i + T] \\ & x(t_i + T) \in S & \end{array}$$

3. Apply to the plant the control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta]$. (the remaining control $\bar{u}(t)$, $t > t_i + \delta$ is discarded)
4. Repeat for the next sampling time $t_i = t_i + \delta$



$\{t_i\}$	Sequence of sampling instants
δ	Sampling interval
x_{t_i}	Plant state measured at time t_i.
$(\bar{x}, \bar{u})$	OL state/control solving the OCP
(x^*, u^*)	CL state/control of the MPC
T, L, W, S	Design parameters

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Necessary Conditions of Optimality: Maximum Principle (smooth version)

If $(\bar{x}, \bar{u})$ solves (P) then $\exists \lambda \in \mathbb{R},\ p \in AC,\ \mu \in C^*$ s.t.

$$\mu\{[0,1]\} + ||p||_{L^\infty} + \lambda > 0,$$

$$-\dot{p}(t) = \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right) \cdot f_x\left(t, \bar{x}(t), \bar{u}(t)\right) \quad \text{a.e. } t \in [0,1],$$

$$-\left(p(1) + \int_{[0,1]} h_x(s, \bar{x}(s))\mu(ds) \right) \in N_C(\bar{x}(1)) + \lambda g_x(\bar{x}(1)),$$

$$\mathrm{supp}\{\mu\} \subset \left\{t \in [0,1] : h\left(t, \bar{x}(t)\right) = 0\right\},$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right) \cdot f\left(t, \bar{x}(t), u\right).$$

The abnormality Phenomenon:

When $\lambda = 0$, the objective function is not taken into account

If $(\bar{x}, \bar{u})$ solves (P) then $\exists \lambda \in \mathbb{R},\ p \in AC,\ \mu \in C^*$ s.t.

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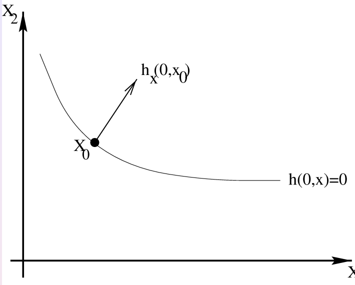
The Degeneracy Phenomenon

Suppose the trajectory starts on the boundary of the admissible region

$$h(0, x_0) = 0,$$

(happens in problems of interest, e.g. Model Predictive Control)



The **Degenerate Multipliers**

$$\lambda = 0, \quad \mu = \delta_{\{0\}}, \quad p = -h_x(0, x_0).$$

(or scalar multiples of these) satisfy the MP for every pair (x, u) we might test. Note that as

$$\lambda = 0, \text{ and } q(t) := \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s)) \mu(ds) \right) = 0 \quad \text{a.e.}$$

the Maximum Principle gives us no information.

The Maximum Principle trivially satisfied for the degenerate multipliers

If $(\bar{x}, \bar{u})$ solves (P) then $\exists \lambda \in \mathbb{R},\ p \in AC,\ \mu \in C^*$ s.t.

$$\mu\{[0,1]\} + ||p||_{L^\infty} + \lambda > 0,$$

$$-\dot{p}(t) = \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right) \cdot f_x\left(t, \bar{x}(t), \bar{u}(t)\right) \quad \text{a.e. } t \in [0,1],$$

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How to avoid Degeneracy?

Ans: Strengthening the MP

For example, strengthening the nontriviality condition to

$$\mu\{(0,1]\} + \lambda + \left\| p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right\|_\infty > 0,$$

- Eliminates only the **Degenerate Multipliers**
- But we have to guarantee that the Strengthened MP is still satisfied for all local minimizers.

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Is the Strengthened MP satisfied for every Problem?

Ans: NO!

Example:[Dubovitskii, in ArutyonovAseev97]

Minimize $x_2(1)$

subject to

$$(\dot{x}_1(t), \dot{x}_2(t)) = (tu(t), u(t))$$
$$(x_1(0), x_2(0)) = (0, 0)$$
$$u(t) \in [-1, 1]$$
$$x_1(t) \geq 0$$

- Here the degenerate multipliers are the only possible choice
- We need condition that identify the problems under which we can Strengthen the MP: **a Constraint Qualification**.

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- Here the degenerate multipliers are the only possible choice
- We need condition that identify the problems under which we can Strengthen the MP: **a Constraint Qualification**.

Outline

Nondegenerate forms of the MP in the literature (< 2010)

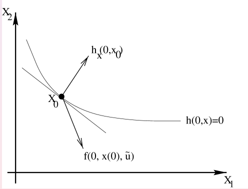
- Early russian references: Dubovitskii,Dubovitskii 85, Arutyunov,Tynianskii85
- Ferreira,Vinter 94 (first reference in English)
- Arutyunov,Aseev 97
- Ferreira,Fontes,Vinter 99
- Arutyunov 2000
- Rampazzo, Vinter 2003
- dePinho,Ferreira,Fontes 2004
- Bettiol,Frankowska 2007
- Lopes, Fontes 2007
- ...

Two groups of Constraint Qualifications

[ArutyunovAseev97, RampazzoVinter03, CerneaFrankowska05, BettiolFrankowska07 ...]

(CQ1) $\exists\delta > 0$, $\exists u$ such that for t near 0

$$h_x(x_0) \cdot f(x_0, u) < 0$$

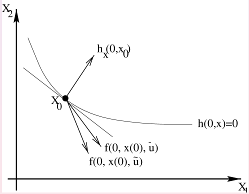


$+$ Not involving the optimal control (easier to verify)
- Typically require more regularity.

[FerreiraVinter94, FerreiraFontesVinter99, dePinhoFerreiraFontes04, LopesFontes07, ...]

(CQ2) $\exists\delta > 0$, $\exists u$ such that for t near 0

$$h_x(x_0) \cdot (f(x_0, u) - f(x_0, \bar{u}(t))) < -\delta$$



- Involving the optimal control (more difficult to verify except in special cases such as CV)
$+$ applicable to wider/less regular class of problems

How do CQ1 and CQ2 relate?

Theorem

[1] *Let* $(\bar{x}, \bar{u})$ *be a local minimizer in the class of piecewise continuous controls. Assume that hypotheses H1-H5 are satisfied. If the trajectory does not leave the boundary immediately then CQ2 implies CQ1.*

[1] Lopes, Sofia O., Fernando ACC Fontes, and Maria do RosÃ¡rio de Pinho. "On constraint qualifications for nondegenerate necessary conditions of optimality applied to optimal control problems." Discrete and Continuous Dynamical Systems (DCDS-A) 29.2 (2011): 559-575.

Constraint Qualification of Integral Type

CQ_I: if $h(0, x_0) = 0$, then there exist positive constants ϵ, ϵ_1, δ and a control function $\hat{u}(t) \in \Omega(t)$ such that for all $t \in [0, \epsilon)$

$$\int_0^t \zeta \cdot [f(\tau, x_0, \hat{u}(\tau)) - f(\tau, x_0, \bar{u}(\tau))] d\tau \leq -\delta t,$$

for all $\zeta \in \partial_x^> h(s, x),\ s \in [0, \epsilon),\ x \in x_0 + \epsilon_1 \mathbb{B}$.

Remark

In this constraint qualification, the inward pointing condition has to be satisfied for some, not all, instants of a neighbourhood of the initial time.

Theorem

[2] *If constraint qualifications* **CQI** *is satified, then a stronger (nondegenerate) version of the maximum principle holds with*

$$\mu\{(0, 1]\} + \|q\|_{L^\infty} + \lambda > 0.$$

[2]S. O. Lopes, F. A. C. C. Fontes, M. d. R. de Pinho, In Integral-type Constraint Qualification to Guarantee Nondegeneracy of the Maximum Principle for Optimal Control Problems with State Constraints, Systems and Control Letters, vol. 62, pp. 686-692, 2013.

Comparison with previous CQ

CQI : if $h(0, x_0) = 0$, then there exist positive constants ϵ, ϵ_1, δ and a control function $\hat{u} \in \mathcal{U}$ such that for all $t \in [0, \epsilon)$

$$\int_0^t h_x(s, x(s)) \cdot [f(\tau, x_0, \hat{u}(\tau)) - f(\tau, x_0, \bar{u}(\tau))] d\tau \leq -\delta t,$$

for all $s \in [0, \epsilon)$, $x \in x_0 + \epsilon_1 \mathbb{B}$.

$\mathbf{CQ_{FFV99}}$: if $h(0, x_0) = 0$, then there exist positive constants ϵ, ϵ_1, δ and a control function $\hat{u} \in \mathcal{U}$ such that for a.e. $t \in [0, \epsilon)$

$$h_x(s, x(s)) \cdot [f(t, x_0, \hat{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta,$$

for all $s \in [0, \epsilon)$, $x \in x_0 + \epsilon_1 \mathbb{B}$.

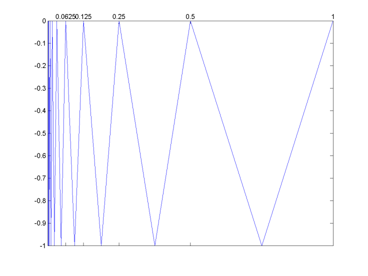
The inequality $h_x \cdot \Delta f < -\delta$ does not have to be satisfied at a.e. points $t \in [0, \epsilon)$, but just on a subset of $t \in [0, \epsilon)$ of positive measure!

Example (in which CQI is satisfied and CQ-ffv is not)

$$
\begin{array}{lll}
\text{Minimize} & -x(1) & \\
\text{subject to} & \dot{x}(t) = u(t) & \text{a.e. } t \in [0,1] \\
& x(0) = 0, \quad x(1) \in \mathbb{R} & \\
& u(t) \in \Omega(t) & \text{a.e. } t \in [0,1] \\
& x(t) \leq 0 & \text{for all } t \in [0,1],
\end{array}
$$

where $\Omega(t) = \{u \in \mathbb{R} : g(t) \leq u \leq 0\}$
and g is the function:

$$
g(t) = \begin{cases} -1, & t = 0 \\ 2 - \frac{4}{2^n} t, & t \in \left[\frac{1}{2}2^{-n}, \frac{3}{4}2^{-n}\right), n \in \\ \frac{4}{2^n} t - 4, & t \in \left[\frac{3}{4}2^{-n}, 2^{-n}\right), n \in \mathbb{N} \end{cases}
$$



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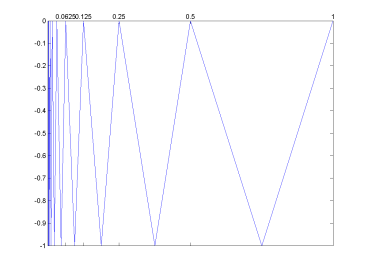

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 & x(0) = 0, \quad x(1) \in \mathbb{R} & \\
 & u(t) \in \Omega(t) & \text{a.e. } \; t \in [0,1] \\
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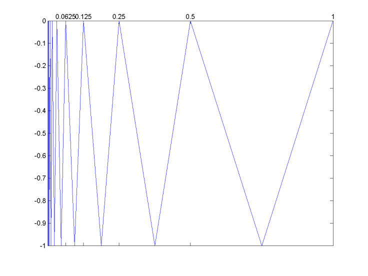

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Example (cont.) I

For this example

$$h_x(s, x(s)) \cdot [f(t, x_0, \hat{u}(t)) - f(t, x_0, \bar{u}(t))] = \hat{u}(t),$$

Thus, the constraint qualifications reduce to:
CQ-ffv: $\exists \delta, \epsilon > 0$ and a control function $\hat{u}$ such that for a.e. $t \in [0, \epsilon)$:

$$\hat{u}(t) < -\delta \quad \text{a.e.} \quad t \in [0, \epsilon) \tag{1}$$

CQI $\exists \delta, \epsilon > 0$ and a control function $\hat{u}$ such that for a.e. $t \in [0, \epsilon)$

$$\int_0^t \hat{u}(\tau) d\tau \leq -\delta t \quad \forall t \in [0, \epsilon), \tag{2}$$

But, for any $\epsilon > 0$, $g(t) = 0$ and $\Omega(t) = \{0\}$ for an infinite number of points $t \in [0, \epsilon)$.
So $\mathbf{CQ_{FFV99}}$ cannot be satisfied.

Example (cont.) II

On the other hand, considering $\hat{u}(t) = g(t)$ then **CQI** is satisfied.
For any $t \in (0, 1]$ we can find an (unique) $k = 2^n$, $n \in \mathbb{N}$ such that

$$\frac{1}{2k} < t \leq \frac{1}{k}.$$

Now

$$\int_0^t g(s)ds = \int_0^{\frac{1}{2k}} g(s)ds + \int_{\frac{1}{2k}}^t g(s)ds.$$

It can be seen that the first term is equal to $-\frac{1}{4k}$and the second term is negative. So, since

$$\int_0^t g(s)ds \leq -\frac{1}{4k} \leq -\frac{1}{4}t,$$

$\mathbf{CQ_I}$ is satisfied.

The inequality in CQ_I just has to be satisfied on *some*, *not all* instants of time in the initial interval.

But, can we identify which are the instants of time in which it has to be satisfied?

Constraint Qualification at outward pointing velocities (F. Fontes, H. Frankowska 2015)

($CQd1_{out}$) $\exists\delta > 0$, $\exists u$ such that

$$h_x(x_0) \cdot [f(t, x_0, u) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta,$$

for a.e. $t \in \{r \in [0, \epsilon] \cup [0, 1] : h_x(\bar{x}(r)) \cdot f(t, \bar{x}(r), \bar{u}(r)) \geq 0\}$.

Remark

The inward pointing inequality just has to be satisfied for the times at which $\bar{x}$ has an outward pointing velocity.

Constraint Qualification at outward pointing velocities

($CQd1_{out}$) $\exists\delta > 0$, $\exists u$ such that

$$h_x(x_0)\cdot f(t, x_0, u) < -\delta,$$

for a.e. $t \in \{r \in [0,\epsilon] \cup [0,1] : h_x(\bar{x}(r)) \cdot f(t, \bar{x}(r), \bar{u}(r)) \geq 0\}$.

$$\Rightarrow$$

($CQd1_{out}$) $\exists\delta > 0$, $\exists u$ such that

$$h_x(x_0)\cdot[f(t, x_0, u) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta,$$

for a.e. $t \in \{r \in [0,\epsilon] \cup [0,1] : h_x(\bar{x}(r)) \cdot f(t, \bar{x}(r), \bar{u}(r)) \geq 0\}$.

Remark

When h is C^1, the constraint qualification $CQ1_{out}$ implies $CQ2_{out}$.

Theorem

[3] *If constraint qualifications CQ_{out} is satisfied, then a stronger (nondegenerate) version of the maximum principle holds with*

$$\mu\{(0,1]\} + \|q\|_{L^\infty} + \lambda > 0.$$

[3] FACC Fontes, H Frankowska, "Normality and Nondegeneracy for Optimal Control Problems with State Constraints", Journal of Optimization Theory and Applications: Volume 166, Issue (2015), Page 115-136

Outline

Normality

$\mathbf{CQ_n}$ (*Constraint Qualification for Normality*)
There exist a positive constants ϵ, δ, K_u, and a control $\hat{u} \in \mathcal{U}$ such that

$$\zeta \cdot [f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta, \tag{3}$$

for all $\zeta \in \partial_x^{>} h(s, \bar{x}(s))$, all $t, s \in (\tau - \epsilon, \tau] \cap [0, 1]$ where τ is defined as

$$\tau = \inf \left\{ t \in [0,1] : \int_{[t,1]} \mu(ds) = 0 \right\}.$$

Theorem

Assume hypotheses ..., $\mathbf{CQ_n}$ and $x(1) \in intC$. Then, the maximum principle is satisfied with $\lambda = 1$. [4]

[4] FACC Fontes, SO Lopes. "Normal Forms of Necessary Conditions for Dynamic Optimization Problems with Pathwise Inequality Constraints". Journal of Mathematical Analysis and Applications, Vol.399 n. 1, pp.27-37, 2013

Constraint qualification for Normality

(CQn_{out}) Define

$$\tau = \inf \left\{ t \in [0,1] : \int_{[t,1]} \mu(ds) = 0 \right\}.$$

$\exists \delta > 0$, $\exists u$ such that

$$h_x(\bar{x}(\tau)) \cdot [f(t,\bar{x}(t),u) - f(t,\bar{x}(t),\bar{u}(t))] < -\delta,$$

for a.e. $t \in \{r \in [\tau - \epsilon, \tau] \cup [0,1] : h_x(\bar{x}(r)) \cdot f(t,\bar{x}(r),\bar{u}(r)) \geq 0\}$.

Theorem

[5] *If constraint qualifications CQn_{out} is satisfied, then a stronger (normal) version of the maximum principle holds with $\lambda + |q(1)| \neq 0$. In particular, if $x(1) \in intC$, then $\lambda = 1$.*

[5] FACC Fontes, H Frankowska, "Normality and Nondegeneracy for Optimal Control Problems with State Constraints", Journal of Optimization Theory and Applications: Volume 166, Issue (2015), Page 115-136

Thank you!
...and happy birthday

60th birthday, Porto

Some references

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