Sampled-Data Model Predictive Control for Nonlinear Time-Varying Systems: Stability and Robustness *

Fernando A. C. C. Fontes[1], Lalo Magni[2], and Éva Gyurkovics[3]

[1] Officina Mathematica, Departamento de Matemática para a Ciência e Tecnologia, Universidade do Minho, 4800-058 Guimarães, Portugal ffontes@mct.uminho.pt
[2] Dipartimento di Informatica e Sistimistica, Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy lalo.magni@unipv.it
[3] Budapest University of Technology and Economics, Institute of Mathematics, Budapest H-1521, Hungary, gye@math.bme.hu

We describe here a sampled-data Model Predictive Control framework that uses continuous-time models but the sampling of the actual state of the plant as well as the computation of the control laws, are carried out at discrete instants of time. This framework can address a very large class of systems, nonlinear, time-varying, and nonholonomic.

As in many others sampled-data Model Predictive Control schemes, Barbalat's lemma has an important role in the proof of nominal stability results. It is argued that the generalization of Barbalat's lemma, described here, can have also a similar role in the proof of robust stability results, allowing also to address a very general class of nonlinear, time-varying, nonholonomic systems, subject to disturbances. The possibility of the framework to accommodate discontinuous feedbacks is essential to achieve both nominal stability and robust stability for such general classes of systems.

1 Introduction

Many Model Predictive Control (MPC) schemes described in the literature use continuous-time models and sample the state of the plant at discrete instants of time. See e.g. [3, 7, 9, 13] and also [6]. There are many advantages in

* The financial support from MURST Project "New techniques for the identification and adaptive control of industrial systems" , from FCT Project POCTI/MAT/61842/2004, and from the Hungarian National Science Foundation for Scientific Research grant no. T037491 is gratefully acknowledged.

considering a continuous-time model for the plant. Nevertheless, any implementable MPC scheme can only measure the state and solve an optimization problem at discrete instants of time.

In all the references cited above, Barbalat's lemma, or a modification of it, is used as an important step to prove stability of the MPC schemes. (Barbalat's lemma is a well-known and powerful tool to deduce asymptotic stability of nonlinear systems, especially time-varying systems, using Lyapunov-like approaches; see e.g. [17] for a discussion and applications). To show that an MPC strategy is stabilizing (in the nominal case), it is shown that if certain design parameters (objective function, terminal set, etc.) are conveniently selected, then the value function is monotone decreasing. Then, applying Barbalat's lemma, attractiveness of the trajectory of the nominal model can be established (i.e. $x(t) \to 0$ as $t \to \infty$). This stability property can be deduced for a very general class of nonlinear systems: including time-varying systems, nonholonomic systems, systems allowing discontinuous feedbacks, etc. If, in addition, the value function possesses some continuity properties, then Lyapunov stability (i.e. the trajectory stays arbitrarily close to the origin provided it starts close enough to the origin) can also be guaranteed (see e.g. [11]). However, this last property might not be possible to achieve for certain classes of systems, for example a car-like vehicle (see [8] for a discussion of this problem and this example).

A similar approach can be used to deduce robust stability of MPC for systems allowing uncertainty. After establishing monotone decrease of the value function, we would want to guarantee that the state trajectory asymptotically approaches some set containing the origin. But, a difficulty encountered is that the predicted trajectory only coincides with the resulting trajectory at specific sampling instants. The robust stability properties can be obtained, as we show, using a generalized version of Barbalat's lemma. These robust stability results are also valid for a very general class of nonlinear time-varying systems allowing discontinuous feedbacks.

The optimal control problems to be solved within the MPC strategy are here formulated with very general admissible sets of controls (say, measurable control functions) making it easier to guarantee, in theoretical terms, the existence of solution. However, some form of finite parameterization of the control functions is required/desirable to solve on-line the optimization problems. It can be shown that the stability or robustness results here described remain valid when the optimization is carried out over a finite parameterization of the controls, such as piecewise constant controls (as in [13]) or as bang-bang discontinuous feedbacks (as in [9]).

2 A Sampled-Data MPC Framework

We shall consider a nonlinear plant with input and state constraints, where the evolution of the state after time t_0 is predicted by the following model.

$$\dot{x}(s) = f(s, x(s), u(s)) \qquad \text{a.e. } s \geq t_0, \tag{1a}$$
$$x(t_0) = x_{t_0} \in X_0, \tag{1b}$$
$$x(s) \in X \subset \mathbb{R}^n \qquad \text{for all } s \geq t_0, \tag{1c}$$
$$u(s) \in U \qquad \text{a.e. } s \geq t_0. \tag{1d}$$

The data of this model comprise a set $X_0 \subset \mathbb{R}^n$ containing all possible initial states at the initial time t_0, a vector x_{t_0} that is the state of the plant measured at time t_0, a given function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a set $U \subset \mathbb{R}^m$ of possible control values.

We assume this system to be asymptotically controllable on X_0 and that for all $t \geq 0$ $f(t, 0, 0) = 0$. We further assume that the function f is continuous and locally Lipschitz with respect to the second argument.

The construction of the feedback law is accomplished by using a sampled-data MPC strategy. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Consider also the control horizon and predictive horizon, T_c and T_p, with $T_p \geq T_c > \delta$, and an auxiliary control law $k^{aux} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T_c, T_p)$ at each sampling instant $t_i \in \pi$, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(t, x_t, T_c, T_p)$: Minimize

$$\int_t^{t+T_p} L(s, x(s), u(s))ds + W(t + T_p, x(t + T_p)), \tag{2}$$

subject to:

$$\dot{x}(s) = f(s, x(s), u(s)) \qquad \text{a.e. } s \in [t, t + T_p], \tag{3}$$
$$x(t) = x_t,$$
$$x(s) \in X \qquad \text{for all } s \in [t, t + T_p],$$
$$u(s) \in U \qquad \text{a.e. } s \in [t, t + T_c],$$
$$u(s) = k^{aux}(s, x(s)) \qquad \text{a.e. } s \in [t + T_c, t + T_p],$$
$$x(t + T_p) \in S. \tag{4}$$

Note that in the interval $[t + T_c, t + T_p]$ the control value is selected from a singleton and therefore the optimization decisions are all carried out in the interval $[t, t + T_c]$ with the expected benefits in the computational time.

The notation adopted here is as follows. The variable t represents real time while we reserve s to denote the time variable used in the prediction model. The vector x_t denotes the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The trajectory is sometimes denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit the dependence on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes our optimal solution to an open-loop optimal control problem. The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the control horizon T_c, the prediction horizon T_p, the running cost and terminal costs functions L and W, the auxiliary control law k^{aux}, and the terminal constraint set $S \subset \mathbb{R}^n$.

The MPC algorithm performs according to a receding horizon strategy, as follows.

1. Measure the current state of the plant $x^*(t_i)$.
2. Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T_c] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x^*(t_i), T_c, T_p)$.
3. Apply to the plant the control $u^*(t) := \bar{u}(t; t_i, x^*(t_i))$ in the interval $[t_i, t_i + \delta)$ (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).
4. Repeat the procedure from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law u^* is a "sampling-feedback" control since during each sampling interval, the control u^* is dependent on the state $x^*(t_i)$. More precisely the resulting trajectory is given by

$$x^*(t_0) = x_{t_0}, \quad \dot{x}^*(t) = f(t, x^*(t), u^*(t)) \quad t \geq t_0,$$

where

$$u^*(t) = k(t, x^*(\lfloor t \rfloor_\pi)) := \bar{u}(t; \lfloor t \rfloor_\pi, x^*(\lfloor t \rfloor_\pi)) \quad t \geq t_0.$$

and the function $t \mapsto \lfloor t \rfloor_\pi$ gives the last sampling instant before t, that is

$$\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}.$$

Similar sampled-data frameworks using continuous-time models and sampling the state of the plant at discrete instants of time were adopted in [2, 7, 8, 6, 13] and are becoming the accepted framework for continuous-time MPC. It can be shown that with this framework it is possible to address —and guarantee stability, and robustness, of the resultant closed-loop system — for a very large class of systems, possibly nonlinear, time-varying and nonholonomic.

3 Nonholonomic Systems and Discontinuous Feedback

There are many physical systems with interest in practice which can only be modelled appropriately as nonholonomic systems. Some examples are the wheeled vehicles, robot manipulators, and many other mechanical systems.

A difficulty encountered in controlling this kind of systems is that any linearization around the origin is uncontrollable and therefore any linear control methods are useless to tackle them. But, perhaps the main challenging characteristic of the nonholonomic systems is that it is not possible to stabilize it if just time-invariant continuous feedbacks are allowed [1]. However, if we allow discontinuous feedbacks, it might not be clear what is the solution of the dynamic differential equation. (See [4, 8] for a further discussion of this issue).

A solution concept that has been proved successful in dealing with stabilization by discontinuous feedbacks for a general class of controllable systems is the concept of "sampling-feedback" solution proposed in [5]. It can be seen that sampled-data MPC framework described can be combined naturally with a "sampling-feedback" law and thus define a trajectory in a way which is very similar to the concept introduced in [5]. Those trajectories are, under mild conditions, well-defined even when the feedback law is discontinuous.

There are in the literature a few works allowing discontinuous feedback laws in the context of MPC. (See [8] for a survey of such works.) The essential feature of those frameworks to allow discontinuities is simply the sampled-data feature — appropriate use of a positive inter-sampling time, combined with an appropriate interpretation of a solution to a discontinuous differential equation.

4 Barbalat's Lemma and Variants

Barbalat's lemma is a well-known and powerful tool to deduce asymptotic stability of nonlinear systems, especially time-varying systems, using Lyapunov-like approaches (see e.g. [17] for a discussion and applications).

Simple variants of this lemma have been used successfully to prove stability results for Model Predictive Control (MPC) of nonlinear and time-varying systems [15, 7]. In fact, in all the sampled-data MPC frameworks cited above, Barbalat's lemma, or a modification of it, is used as an important step to prove stability of the MPC schemes. It is shown that if certain design parameters (objective function, terminal set, etc.) are conveniently selected, then the value function is monotone decreasing. Then, applying Barbalat's lemma, attractiveness of the trajectory of the nominal model can be established (i.e. $x(t) \to 0$ as $t \to \infty$). This stability property can be deduced for a very general class of nonlinear systems: including time-varying systems, nonholonomic systems, systems allowing discontinuous feedbacks, etc.

A recent work on robust MPC of nonlinear systems [9] used a generalization of Barbalat's lemma as an important step to prove stability of the algorithm. However, it is our believe that such generalization of the lemma might provide a useful tool to analyse stability in other robust continuous-time MPC approaches, such as the one described here for time-varying systems.

A standard result in Calculus states that if a function is lower bounded and decreasing, then it converges to a limit. However, we cannot conclude whether its derivative will decrease or not unless we impose some smoothness property on $\dot{f}(t)$. We have in this way a well-known form of the Barbalat's lemma (see e.g. [17]).

Lemma 1 (Barbalat's lemma 1). *Let $t \mapsto F(t)$ be a differentiable function with a finite limit as $t \to \infty$. If $\dot{F}$ is uniformly continuous, then $\dot{F}(t) \to 0$ as $t \to \infty$.*

A simple modification that has been useful in some MPC (nominal) stability results [15, 7] is the following.

Lemma 2 (Barbalat's lemma 2). *Let M be a continuous, positive definite function and x be an absolutely continuous function on $\mathbb{R}$. If $\|x(\cdot)\|_{L^\infty} < \infty$, $\|\dot{x}(\cdot)\|_{L^\infty} < \infty$, and $\lim_{T\to\infty} \int_0^T M(x(t))\,dt < \infty$, then $x(t) \to 0 \quad as \quad t \to \infty$.*

Now, suppose that due to disturbances we have no means of guaranteeing that all the hypothesis of the lemma are satisfied for the trajectory x^* we want to analyse. Instead some hypothesis are satisfied on a neighbouring trajectory $\hat{x}$ that coincides with the former at a sequence of instants of time. [4] Furthermore, suppose that instead of approaching the origin we would like to approach some set containing the origin. These are the conditions of the following lemma.

Definition 1. *Let A be a nonempty, closed subset of $\mathbb{R}^n$. The function $x \mapsto d_A(x)$, from $\mathbb{R}^n$ to $\mathbb{R}$, denotes the distance from a point x to the set A (i.e. $d_A(x) := \min_{y\in A} \|x - y\|$).*

We say that a function M is positive definite with respect to the set A *if $M(x) > 0$ for all $x \notin A$ and $M(x) = 0$ for some $x \in A$.*

Lemma 3 (A generalization of Barbalat's lemma). *Let A be subset of $\mathbb{R}^n$ containing the origin, and $M : \mathbb{R}^n \to \mathbb{R}$ be a continuous function which is positive definite with respect to A.*

Let $\Delta > 0$ be given and for any $\delta \in (0, \Delta)$ consider the functions x^*_δ and $\hat{x}_\delta$ from $\mathbb{R}^+$ to $\mathbb{R}^n$ satisfying the following properties:

- *The function x^*_δ is absolutely continuous, the function $\hat{x}_\delta$ is absolutely continuous on each interval $[i\delta, (i+1)\delta)$, for all $i \in \mathbb{N}_0$, and $\hat{x}_\delta(i\delta) = x^*_\delta(i\delta)$ for all $i \in \mathbb{N}_0$.*
- *There exist positive constants K_1, K_2 and K_3 such that for all $\delta \in (0, \Delta)$*

$$\|\dot{x}^*_\delta(\cdot)\|_{L^\infty(0,\infty)} < K_1, \quad \|\hat{x}_\delta(\cdot)\|_{L^\infty(0,\infty)} < K_2, \quad \|\dot{\hat{x}}_\delta(\cdot)\|_{L^\infty(0,\infty))} < K_3.$$

Moreover,

$$\lim_{\tau\to\infty} \int_0^\tau M(\hat{x}_\delta(t))\,dt < \infty, \tag{5}$$

Then for any $\epsilon > 0$ there is a $\delta(\epsilon)$ and for any $\delta \in (0, \delta(\epsilon))$ there is a $T = T(\epsilon, \delta) > 0$ such that

$$d_A(\hat{x}_\delta(t)) \le \epsilon, \quad d_A(x^*_\delta(t)) \le \epsilon \quad \text{for all } t \ge T. \tag{6}$$

[4] In an NMPC context, $\hat{x}$ would represent the concatenation of predicted trajectories; see equation (12)

4.1 Proof of Lemma 3

First we shall show that the statement is true for the parameterized family $\widehat{x}_\delta(.)$. Suppose the contrary: there exists an ε_0 such that for any $\overline{\delta} \in (0, \Delta]$ – thus for $\overline{\delta} = \min\{\Delta, \varepsilon_0/(2K_3)\}$, as well – one can show a $\widetilde{\delta} \in (0, \overline{\delta}]$ and a sequence $\{t_k\}_{k=0}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$ so that $d_A(\widehat{x}_\delta(t_k)) > \varepsilon_0$ for all $k \in \mathbb{N}$.

Without loss of generality, we may assume that $t_{k+1} - t_k \geq \widetilde{\delta}$, $k \in \mathbb{N}$. Let $R \geq K_2$ be such that the set $B = \{x \in \mathbb{R}^n : \ d_A(x)) \geq \varepsilon_0/2 \text{ and } \|x\| \leq R\}$ is nonempty. Since B is compact, M is continuous and $M(x) > 0$, if $x \in B$, there exists an $m > 0$ such that $m \leq M(x)$ for all $x \in B$. Since $t_k \to \infty$ as $k \to \infty$, for any $k \in \mathbb{N}$ one can show a j_k so that $t_k \in [j_k\widetilde{\delta},\, (j_k+1)\widetilde{\delta})$. Note that for any $t \in [j_k\widetilde{\delta},\, (j_k+1)\widetilde{\delta})$ we have

$$\left\|\widehat{x}_{\widetilde{\delta}}(t) - \widehat{x}_{\widetilde{\delta}}(t_k)\right\| \leq \int_{t_k}^{t} \left\|\dot{\widehat{x}}_{\widetilde{\delta}}(s)\right\| ds \leq K_3\widetilde{\delta} \leq \varepsilon_0/2.$$

Then, by the triangle inequality

$$d_A(\widehat{x}_{\widetilde{\delta}}(t)) \geq d_A(\widehat{x}_{\widetilde{\delta}}(t_k)) - \left\|\widehat{x}_{\widetilde{\delta}}(t) - \widehat{x}_{\widetilde{\delta}}(t_k)\right\| \geq \varepsilon_0/2.$$

Therefore $\widehat{x}_{\widetilde{\delta}}(t) \in B$, if $t \in [j_k\widetilde{\delta},\, (j_k+1)\widetilde{\delta})$, thus

$$\int_{j_k\widetilde{\delta}}^{(j_k+1)\widetilde{\delta}} M(\widehat{x}_{\widetilde{\delta}}(s))ds \geq m\widetilde{\delta}.$$

This would imply that $\lim_{\tau\to\infty} \int_0^{\tau} M(\widehat{x}_\delta\,(s))ds \to \infty$ contradicting (5).

Now let $\varepsilon > 0$ be arbitrarily given, let $\delta_1 := \varepsilon/(2K_1)$, and let $\varepsilon_1 := \varepsilon/2$. From the preceding part of the proof we already know that there is a $\delta_2 = \widehat{\delta}(\varepsilon_1)$, and for any $0 < \delta < \min\{\delta_1, \delta_2\}$ there is a $\widehat{T}(\varepsilon_1, \delta)$ such that

$$d_A(\widehat{x}_\delta(t)) \leq \varepsilon_1 \qquad \text{for all } t \geq \widehat{T}(\varepsilon_1, \delta).$$

On the other hand, if $t \geq \widehat{T}(\varepsilon_1, \delta)$ is arbitrary but fixed, then $t \in [i\delta,\, (i+1)\delta)$ for some i, thus by the triangle inequality and by the assumptions of the lemma we have

$$d_A(x^*_\delta(t)) \leq d_A(x^*_\delta(i\delta)) + \|x^*_\delta(t) - x^*_\delta(i\delta)\| \leq \varepsilon_1 + K_1\delta \leq \varepsilon.$$

Therefore $\delta(\varepsilon) = \min\{\delta_1, \delta_2\}$ and $T(\varepsilon, \delta) = \widehat{T}(\varepsilon_1, \delta)$ are suitable.□

5 Nominal Stability

A stability analysis can be carried out to show that if the design parameters are conveniently selected (i.e. selected to satisfy a certain sufficient stability condition, see e.g. [7]), then a certain MPC value function V is shown to be monotone decreasing. More precisely, for some $\delta > 0$ small enough and for any $t'' > t' > 0$

$$V(t'', x^*(t'')) - V(t', x^*(t')) \leq -\int_{t'}^{t''} M(\hat{x}(s))ds. \tag{7}$$

where M is a continuous, radially unbounded, positive definite function. The MPC value function V is defined as

$$V(t, x) := V_{\lfloor t \rfloor_\pi}(t, x)$$

where $V_{t_i}(t, x_t)$ is the value function for the optimal control problem $\mathcal{P}(t, x_t, T_c - (t - t_i), T_c - (t - t_i))$ (the optimal control problem defined where the horizon is shrank in its initial part by $t - t_i$).

From (7) we can then write that for any $t \geq t_0$

$$0 \leq V(t, x^*(t)) \leq V(t_0, x^*(t_0)) - \int_{t_0}^{t} M(x^*(s))ds.$$

Since $V(t_0, x^*(t_0))$ is finite, we conclude that the function $t \mapsto V(t, x^*(t))$ is bounded and then that $t \mapsto \int_{t_0}^{t} M(x^*(s))ds$ is also bounded. Therefore $t \mapsto x^*(t)$ is bounded and, since f is continuous and takes values on bounded sets of (x, u), $t \mapsto \dot{x}^*(t)$ is also bounded. All the conditions to apply Barbalat's lemma 2 are met, yielding that the trajectory asymptotically converges to the origin. Note that this notion of stability does not necessarily include the Lyapunov stability property as is usual in other notions of stability; see [8] for a discussion.

6 Robust Stability

In the last years the synthesis of robust MPC laws is considered in different works [14].

The framework described below is based on the one in [9], extended to time-varying systems.

Our objective is to drive to a given target set Θ ($\subset \mathbb{R}^n$) the state of the nonlinear system subject to bounded disturbances

$$\dot{x}(t) = f(t, x(t), u(t), d(t)) \qquad \text{a.e. } t \geq t_0, \tag{8a}$$
$$x(t_0) = x_0 \in X_0, \tag{8b}$$
$$x(t) \in X \qquad \text{for all } t \geq t_0, \tag{8c}$$
$$u(t) \in U \qquad \text{a.e. } t \geq t_0, \tag{8d}$$
$$d(t) \in D \qquad \text{a.e. } t \geq t_0, \tag{8e}$$

where $X_0 \subset \mathbb{R}^n$ is the set of possible initial states, $X \subset \mathbb{R}^n$ is the set of possible states of the trajectory, $U \subset \mathbb{R}^m$ is a bounded set of possible control values, $D \subset \mathbb{R}^p$ is a bounded set of possible disturbance values, and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is a given function. The state at time t from the trajectory x, starting from x_0 at t_0, and solving (8) is denoted $x(t; t_0, x_0, u, d)$ when we want to make explicit the dependence on the initial state, control and disturbance. It is also convenient to define, for $t_1,\ t_2 \geq t_0$, the function spaces

$$\mathcal{U}([t_1, t_2]) := \{u : [t_1, t_2] \to \mathbb{R}^m : u(s) \in U,\ s \in [t_1, t_2]\},$$
$$\mathcal{D}([t_1, t_2]) := \{d : [t_1, t_2] \to \mathbb{R}^p : d(s) \in D,\ s \in [t_1, t_2]\}.$$

The target set Θ is a closed set, contains the origin and is robustly invariant under no control. That is, $x(t; t_0, x_0, u, d) \in \Theta$ for all $t \geq t_0$, all $x_0 \in \Theta$, and all $d \in \mathcal{D}([t_0, t))$ when $u \equiv 0$. We further assume that f is a continuous function and locally Lipschitz continuous with respect to x.

Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with constant inter-sampling times $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Let the control horizon T_c and prediction horizon T_p, with $T_c \leq T_p$, be multiples of δ ($T_c = N_c\delta$ and $T_p = N_p\delta$ with $N_c, N_p \in \mathbb{N}$). Consider also a terminal set S ($\subset \mathbb{R}^n$), a terminal cost function $W : \mathbb{R}^n \to \mathbb{R}$ and a running cost function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. The optimization problem is a finite horizon differential game where the disturbance d acts as the maximizing player and the control u acts as the minimizing player. We shall assume that the minimizing player uses a sampled-data information structure. The space of the corresponding strategies over $[t_1, t_2]$ we denote by $\mathcal{K}([t_1, t_2])$. For any $t \in \pi$, let $k_t^{aux} \in \mathcal{K}([t+T_c, t+T_p])$ be an a priori given auxiliary sampled-data strategy. The quantities time horizons T_c and T_p, objective functions L and W, terminal constraint set S, the inter-sampling time δ, and auxiliary strategy k_t^{aux} are the quantities we are able to tune — the so-called *design parameters* — and should be chosen to satisfy the robust stability condition described below.

At a certain instant $t \in \pi$, we select for the prediction model the control strategy for the intervals $[t, t+T_c)$ and $[t+T_c, t+T_p)$ in the following way. In the interval $[t, t+T_c)$, we should select, by solving an optimization problem, the strategy k_t in the interval $[t, t+T_c]$. The strategy k_t^{aux}, known a priori, is used in the interval $[t+T_c, t+T_p]$.

The robust feedback MPC strategy is obtained by repeatedly solving on-line, at each sampling instant t_i, a min-max optimization problem $\mathcal{P}$, to select the feedback k_{t_i}, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(t, x_t, T_c, T_p)$: $\text{Min}_{k \in \mathcal{K}([t,t+T_c])} \text{Max}_{d \in \mathcal{D}([t,T_p])}$

$$\int_t^{t+T_p} L(x(s), u(s))ds + W(x(t+T_p)) \tag{9}$$

subject to:

$$
\begin{aligned}
&x(t) = x_t && \\
&\dot{x}(s) = f(s, x(s), u(s), d(s)) && \text{a.e. } s \in [t, t+T_p] \qquad (10)\\
&x(s) \in X && \text{for all } s \in [t, t+T_p] \\
&u(s) \in U && \text{a.e. } s \in [t, t+T_p] \\
&x(t+T_p) \in S, && \qquad (11)
\end{aligned}
$$

where

$$u(s) = k_t(s, x(\lfloor s \rfloor_\pi)) \quad \text{for } s \in [t, t+T_c)$$

$$u(s) = k_t^{aux}(s, x(\lfloor s \rfloor_\pi)) \quad \text{for} \quad s \in [t+T_c, t+T_p).$$

In this optimization problem we use the convention that if some of the constraint is not satisfied, then the value of the game is $+\infty$. This ensures that when the value of the game is finite, the optimal control strategy guarantees the satisfaction of the constraints for all possible disturbance scenarios.

The MPC algorithm performs according to a Receding Horizon strategy, as follows:

1. Measure the current state of the plant $x^*(t_i)$.
2. Compute the feedback k_{t_i}, solution to problem $\mathcal{P}(t_i, x^*(t_i), T_c, T_p)$.
3. Apply to the plant the control given by the feedback law k_{t_i} in the interval $[t_i, t_i + \delta)$, (discard all the remaining data for $t \geq t_i + \delta$).
4. Repeat the procedure from (1.) for the next sampling instant t_{i+1}.

The main stability result states that if the design parameters are chosen to satisfy the robust stability conditions RSC, then the MPC strategy ensures steering to a certain target set Θ. The following definitions will be used.

Definition 2. *The sampling-feedback k is said to* robustly stabilize *the system to the target set Θ if for any $\epsilon > 0$ there exists a sufficiently small inter-sample time δ such that we can find a scalar $T > 0$ satisfying $d_\Theta(x(t)) \leq \epsilon$ for all $t \geq T$.*

Definition 3. *The* playable set *$\Omega(t, T_c, T_p, S)$ is the set of all initial states x_t for which using the inter-sampling time $\delta \in (0, \Delta]$ and the auxiliary strategy k_t^{aux} there exists some control strategy $k_t \in \mathcal{K}$ for $[t, t+T_p]$ with $k_t(s,.) = k_t^{aux}(s,.)$ for $s \in [t+T_c, t+T_p]$ such that $x(t+T_p; t, x_t, k_t, d) \in S$ for all $d \in \mathcal{D}([t, t+T_p])$.*

Consider the following robust stability condition

RSC: The design parameters: time horizons T_c and T_p, objective functions L and W, terminal constraint set S, inter-sampling time δ, and auxiliary feedback strategy k_t^{aux} satisfy

RSC1 The set S is closed, contains the origin, and is contained in X. Also $k_t^{aux}(s, x) \in U$ for all $x \in X$, $s \in [t+T_c, t+T_p]$ and $t \in \pi$.

RSC2 The function L is continuous, $L(0,0) = 0$, and for all $u \in U$ we have that $L(x,u) \geq M(x)$ for some function $M : \mathbb{R}^n \to \mathbb{R}_+$ which is continuous, radially unbounded and positive definite with respect to the set Θ.

RSC3 The function W is Lipschitz continuous and $W(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

RSC4 The set of initial states X_0 is contained in the playable set $\Omega(t_0, T_c, T_p, S)$.

RSC5 For each sampling instant $t \in \pi$ and each $x_t \in S \backslash \Theta$, and for all possible disturbances $d \in \mathcal{D}([t, t+\delta)])$, using the notation $x(s) = x(s; t, x_t, k_t^{aux}, d)$, we have

$$W(x(t+\delta)) - W(x_t) \leq -\int_t^{t+\delta} L(x(s), k_t^{aux}(s, x_t))\ \mathrm{d}s \qquad (RSC5a)$$

$$x(t+\delta) \in S. \qquad (RSC5b)$$

We are in the conditions to state the following stability result.

Theorem 1. *Assume condition RSC is satisfied and that the differential games $\mathcal{P}(t, x_t, T_c, T_p)$ have a value for all $x_t \in X$ and $t \geq t_0$. Then, the robust MPC strategy robustly stabilizes the system to the target set Θ.*

Proof. The proof starts by establishing a monotone decreasing property of the MPC Value function. Then the application of the generalized Barbalat's Lemma yields the robust stability result.

Let $V(t,x)$ be the value function of $\mathcal{P}(t, x, T_c - (t - t_i), T_p - (t - t_i))$ with $t_i = \lfloor t \rfloor_\pi$. Let also $\hat{x}$ be the concatenation of predicted trajectories $\bar{x}$ for each optimization problem. That is for $i \geq 0$

$$\hat{x}(t) = \bar{x}^i(t) \quad \text{for all } t \in [t_i, t_i + \delta) \qquad (12)$$

where $\bar{x}^i$ is the trajectory of a solution to problem $\mathcal{P}(t_i, x^*(t_i), T_c, T_p)$. Note that $\hat{x}$ coincides with x^* at all sampling instants $t_i \in \pi$, but they are typically not identical on $[t_i, t_i + \delta)$, since they correspond to different disturbances.

The following lemma establishes a monotone decreasing property of V .

Lemma 4. *([9], Lemma 4.4) There exists an inter-sample time $\delta > 0$ small enough such that for any $t' < t''$, if $x^*(t''), x^*(t') \notin \Theta$ then*

$$V(t'', x^*(t'')) - V(t', x^*(t')) \leq -\int_{t'}^{t''} M(\hat{x}(s)) ds.$$

We can then write that for any $t \geq t_0$

$$0 \leq V(t, x^*(t)) \leq V(t_0, x^*(t_0)) - \int_{t_0}^{t} M(\hat{x}(s)) ds.$$

Since $V(t_0, x^*(t_0))$ is finite, we conclude that the function $t \mapsto V(t, x^*(t))$ is bounded and then that $t \mapsto \int_{t_0}^{t} M(\hat{x}(s))ds$ is also bounded. Therefore $t \mapsto \hat{x}(t)$ is bounded and, since f is continuous and takes values on bounded sets of (x, u, d), $t \mapsto \dot{\hat{x}}(t)$ is also bounded. Using the fact that x^* is absolutely continuous and coincides with $\hat{x}$ at all sampling instants, we may deduce that $t \mapsto \dot{x}^*(t)$ and $t \mapsto x^*(t)$ are also bounded. We are in the conditions to apply the previously established Generalization of Barbalat's Lemma 3, yielding the assertion of the theorem.

7 Finite Parameterizations of the Control Functions

The results on stability and robust stability were proved using an optimal control problem where the controls are functions selected from a very general set (the set of measurable functions taking values on a set U, subset of R^m). This is adequate to prove theoretical stability results and it even permits to use the results on existence of a minimizing solution to optimal control problems (e.g. [7, Proposition 2]). However, for implementation, using any optimization algorithm, the control functions need to be described by a finite number of parameters (the so called finite parameterizations of the control functions). The control can be parameterized as piecewise constant controls (e.g. [13]), polynomials or splines described by a finite number of coeficients, bang-bang controls (e.g. [10, 9]), etc. Note that we are not considering discretization of the model or the dynamic equation. The problems of discrete approximations are discussed in detail e.g. in [16] and [12].

But, in the proof of stability, we just have to show at some point that the optimal cost (the value function) is lower than the cost of using another *admissible* control. So, as long as the set of admissible control values U is constant for all time, an easy, but nevertheless important, corollary of the previous stability results follows

> *If we consider the set of admissible control functions (including the auxiliary control law) to be a finitely parameterizable set such that the set of admissible control values is constant for all time, then both the nominal stability and robust stability results here described remain valid.*

An example, is the use of discontinuous feedback control strategies of bang-bang type, which can be described by a small number of parameters and so make the problem computationally tractable. In bang-bang feedback strategies, the controls values of the strategy are only allowed to be at one of the extremes of its range. Many control problems of interest admit a bang-bang stabilizing control. Fontes and Magni [9] describe the application of this parameterization to a unicycle mobile robot subject to bounded disturbances.

References

1. R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. S. Sussmann, editors, *Differential Geometric Control Theory*, pages 181–191. Birkhouser, Boston, 1983.
2. H. Chen and F. Allgöwer. Nonlinear model predictive control schemes with guaranteed stability. In R. Berber and C. Kravaris, editors, *Nonlinear Model Based Process Control*. Kluwer, 1998.
3. H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1217, 1998.
4. F. H. Clarke. Nonsmooth analysis in control theory: a survey. *European Journal of Control; Special issue: Fundamental Issues in Control*, 7:145–159, 2001.
5. F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10):1394–1407, 1997.
6. R. Findeisen, L. Imsland, F. Allgower, and B. Foss. Towards a sampled-data theory for nonlinear model predictive control. In W. Kang, M. Xiao, and C. Borges, editors, *New Trends in Nonlinear Dynamics and Control, and their applications*, volume 295 of *Lecture Notes in Control and Information Sciences*, pages 295–311. Springer Verlag, Berlin, 2003.
7. F. A. C. C. Fontes. A general framework to design stabilizing nonlinear model predictive controllers. *Systems & Control Letters*, 42:127–143, 2001.
8. F. A. C. C. Fontes. Discontinuous feedbacks, discontinuous optimal controls, and continuous-time model predictive control. *International Journal of Robust and Nonlinear Control*, 13(3–4):191–209, 2003.
9. F. A. C. C. Fontes and L. Magni. Min-max model predictive control of nonlinear systems using discontinuous feedbacks. *IEEE Transactions on Automatic Control*, 48:1750–1755, 2003.
10. L. Grüne. Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal of Control and Optimization*, 98(4):1288–1308, 2000.
11. E. Gyurkovics. Receding horizon control via Bolza-time optimization. *Systems and Control Letters*, 35:195–200, 1998.
12. E. Gyurkovics and A. M. Elaiw. Stabilization of smpled-data nonlinear systems by receding horizon control via discrete-time aproximations. *Automatica*, 40:2017–2028, 2004.
13. L. Magni and R. Scattolini. Model predictive control of continuous-time nonlinear systems with piecewise constant control. *IEEE Transactions on Automatic Control*, 49:900–906, 2004.
14. L. Magni and R. Scattolini. Robustness and robust design of mpc for nonlinear discrete-time systems. In *Preprints of NMPC05 - International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, pages 31–46, IST, University of Stuttgart, August 26-30, 2005.
15. H. Michalska and R. B. Vinter. Nonlinear stabilization using discontinuous moving-horizon control. *IMA Journal of Mathematical Control and Information*, 11:321–340, 1994.
16. D. Nešić and A. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control*, 49:1103–1034, 2004.
17. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, New Jersey, 1991.

Appendix

7.1 Proof of Lemma 4

At a certain sampling instant t_i, we measure the current state of the plant x_{t_i} and we solve problem $\mathcal{P}(x_{t_i}, T_c, T_p)$ obtaining as solution the feedback strategy $\bar{k}$ to which corresponds, in the worst disturbance scenario, the trajectory $\bar{x}$ and control $\bar{u}$. The value of differential game $\mathcal{P}(x_{t_i}, T_c, T_p)$ is given by

$$V_{t_i}(t_i, x_{t_i}) = \int_{t_i}^{t_i+T_p} L(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t_i + T_p)). \tag{13}$$

Consider now the family of problems $\mathcal{P}(x_t, T_c - (t - t_i), T_p - (t - t_i))$ for $t \in [t_i, t_i + \delta)$. These problems start at different instants t, but all terminate at the same instant $t_i + T_p$. Therefore in the worst disturbance scenario, by Bellman's principle of optimality we have that

$$V_{t_i}(t, \bar{x}(t)) = \int_{t}^{t_i+T_p} L(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t_i + T_p)). \tag{14}$$

Suppose that the worst disturbance scenario did not occur and so, at time t, we are at state $x^*(t)$ which is, in general, distinct from $\bar{x}(t)$. Because such scenario is more favorable, and by the assumption on the existence of value to the differential game we have that

$$V_{t_i}(t, x^*(t)) \leq V_{t_i}(t, \bar{x}(t)) \qquad \text{for all } t \in [t_i, t_i + \delta). \tag{15}$$

We may remove the subscript t_i from the value function if we always choose the subscript t_i to be the sampling instant immediately before t, that is (recall that $\lfloor t \rfloor_\pi = \max_i \{t_i \in \pi : t_i \leq t\}$)

$$V(t, x) := V_{\lfloor t \rfloor_\pi}(t, x).$$

For simplicity define the function

$$V^*(t) = V(t, x^*(t)).$$

We show that $t \mapsto V^*(t)$ is decreasing in two situations:

(i) on each interval $[t_i, t_i + \delta)$

$$V^*(t) \leq V^*(t_i) - \int_{t_i}^{t} M(\bar{x}(s))\, ds \quad \text{for all } t \in [t_i, t_i + \delta), \text{ and all } i \geq 0;$$

(ii) from one interval to the other

$$V^*(t_i + \delta^+) \leq V^*(t_i + \delta^-) \quad \text{for all } i \geq 0;$$

therefore yielding the result.

(i) The first assertion is almost immediate from (15), (14) and (13).

$$\begin{aligned} V^*(t) &\le V_{t_i}(t, \bar{x}(t)) \\ &= V_{t_i}(t_i, \bar{x}(t_i)) - \int_{t_i}^{t} L(\bar{x}(s), \bar{u}(s))\, ds \\ &= V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^{t} L(\bar{x}(s), \bar{u}(s))\, ds \\ &\le V^*(t_i) - \int_{t_i}^{t} M(\bar{x}(s))\, ds \end{aligned}$$

for all $t \in [t_i, t_i + \delta)$, and all $i \ge 0$;

(ii) Let the pair $(\bar{x}, \bar{u})|_{[t_i+\delta^-, t_i+T_p]}$ be an optimal solution to $\mathcal{P}(x(t_i+\delta^-), T_c - \delta, T_p - \delta)$. Then

$$\begin{aligned} V^*(t_i + \delta^-) &= V_{t_i}(t_i + \delta^-, \bar{x}(t_i + \delta^-)) \\ &= \int_{t_i+\delta}^{t_i+T_p} L(\bar{x}(s), \bar{u}(s))\, ds + W(\bar{x}(t_i + T_p)). \end{aligned}$$

Now, extend the process $(\bar{x}, \bar{u})$ to the interval $[t_i + \delta^-, t_i + T_p + \delta^-]$ using in the last δ seconds the auxiliary control law k^{aux}. This is an admissible (the set S is invariant under k^{aux} and is contained in X), suboptimal solution to $\mathcal{P}(x(t_i + \delta^+), T_c - \delta, T_p - \delta)$. Therefore

$$\begin{aligned} V^*(t_i + \delta^+) &= V_{t_{i+1}}(t_i + \delta^+, \bar{x}(t_i + \delta^+)) \\ &\le \int_{t_i+\delta}^{t_i+T_p+\delta} L(\bar{x}(s), \bar{u}(s))\, ds + W(\bar{x}(t_i + T_p + \delta)) \\ &= V^*(t_i + \delta^-) + W(\bar{x}(t_i + T_p + \delta)) - W(\bar{x}(t_i + T_p)) \\ &\qquad + \int_{t_i+T_p}^{t_i+T_p+\delta} L(\bar{x}(s), \bar{u}(s))\, ds. \end{aligned}$$

Using RSC5a in the interval $[t_i + T_p, t_i + T_p + \delta]$ we obtain

$$V^*(t_i + \delta^+) \le V^*(t_i + \delta^-) \quad \text{for all } i \ge 0$$

as required.