Min-Max Model Predictive Control of Nonlinear Systems using Discontinuous Feedbacks

Fernando A. C. C. Fontes and Lalo Magni

Abstract

This paper proposes a Model Predictive Control (MPC) algorithm for the solution of a robust control problem for continuous-time systems. Discontinuous feedback strategies are allowed in the solution of the min-max problems to be solved. The use of such strategies allows MPC to address a large class of nonlinear systems, including among others nonholonomic systems. Robust stability conditions to ensure steering to a certain set under bounded disturbances are established. The use of bang-bang feedbacks described by a small number of parameters is proposed, reducing considerably the computational burden associated with solving a differential game. The applicability of the proposed algorithm is tested to control a unicycle mobile robot.

keywords: Predictive control; receding horizon; robust synthesis, discontinuous feedbacks.

I. Introduction

In this work we address the problem of synthesizing a discontinuous feedback law to stabilize a constrained nonlinear system subject to bounded disturbances.

It is well-known that there is a class of nonlinear systems (including some with interest in practice, such as nonholonomic systems) that cannot be stabilized by a smooth (C^1) feedback law [1], [2]. Despite that, there are not many constructive design methods to generate nonsmooth stabilizing feedback laws. See, for example, the survey [3] and references therein (having an emphasis on backstepping methods that are limited to systems in triangular form), the work [4] using continuous (though nonsmooth) feedbacks, the work [5] transforming the system into a discontinuous one, the work [6] addressing homogeneous systems, among a few others. Regarding frameworks that additionally deal explicitly with some form of uncertainty, the number of existing methods is even more reduced. We mention, as example, methods based on constructing robust control Lyapunov functions [7], and adaptive methods for systems with parametric uncertainty [8]. If, in addition, we allow the system to have input constraints, pathwise constraints, and be subject to bounded disturbances, then we are not aware of any general constructive methodology to generate stabilizing feedbacks having been previously reported in literature.

The technique used here is based on the Model Predictive Control (MPC) concept, also known as Receding Horizon Control. Generally speaking, the feedback control law is constructed by solving on-line a sequence of dynamic optimization problems, each of them using the current (measured) state of the plant.

Model predictive control is an increasingly popular control technique. It has been widely used in industry: it has been classified as the only control technique with a substantial impact on industrial control [9]. MPC has also been witnessing in the recent years a considerable interest of the research community and, consequently, important theoretical developments (see e.g. the survey [10]). This success can be explained by the fact that, similarly to optimal control, MPC has an inherent ability to deal naturally with constraints on the inputs and on the state. Moreover, since the controls generated are closed-loop strategies obtained by optimizing some criterion, the method possesses some desirable performance properties and also intrinsic robustness properties [11]. The applicability of the MPC method to continuous-time systems

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Officina Mathematica, Departamento de Matemática para a Ciência e Tecnologia, Universidade do Minho, 4800-058 Guimarães, Portugal (e-mail:ffontes@mct.uminho.pt).

Dipartimento di Informatica e Sistimistica, Università degli Studi di Pavia, Italy. (e-mail: lalo.magni@unipv.it).

has recently been extended to admit discontinuous feedbacks [12], allowing MPC to address a large class of nonlinear systems, including nonholonomic systems. Remarkably, as it is underlined in [3], and thanks to the recent extensions, MPC is the only general method for calculating stabilizing feedbacks in the absence of an explicit Lyapunov function.

In the last years the synthesis of robust MPC laws is considered in different works. See e.g. [13] where the robust problem is solved for nonlinear continuous-time systems if smooth control laws are considered. Guaranteeing robustness of MPC is even more important when discontinuous feedbacks are allowed because, in that case, problems of lack of robustness might arise, as shown recently in [14]. A continuous-time MPC framework generating discontinuous robust control laws is, to the best of our knowledge, a novelty of the present work.

In robust MPC approaches, as the one reported here, the dynamic optimization problems to be solved are min-max optimal control problems. A keystone in such frameworks, that is now becoming accepted, is that the optimal control problems should search for feedback strategies and not open-loop controls. The open-loop min-max MPC may be very conservative. It is often unrealistic to presume that a unique open-loop control function would lead to the expected behaviour in all possible disturbance situations. This may lead to low performance solutions (the value of a feedback min-max optimization problem is always lower than the value of the corresponding open-loop min-max optimization problem; see [15]) and even unfeasibility problems [16], [10], [17].

However, the optimization problem of finding a feedback strategy is considerably more complex than the problem of finding an open-loop control function. (The high complexity remains even when using the equivalent formulation of searching for non-anticipative strategies for the minimizing player [18].) Thus, most of the "feedback MPC" methods reported have been considered more conceptual rather than practical. To make computations viable the feedback strategies sought for must be parameterized in same way. In this respect, we investigate here the use of discontinuous feedback control strategies of bang-bang type, which can be described by a small number of parameters and so make the problem computationally tractable.

In bang-bang feedback strategies, the controls values of the strategy are only allowed to be at one of the extremes of its range. Many control problems of interest admit a bang-bang stabilizing control. These include some input constrained problems in the process industry, some nonholonomic systems which frequently arise in robotics and other applications. Examples of such nonholonomic systems are the unicycle system investigated below, and the Brockett integrator addressed by bang-bang in [6]. Bang-bang control is, for example, the solution in optimal control of linear systems when the optimality criterion is linear or minimum-time [19]. The problem of finding an optimal bang-bang feedback control can be equivalently stated as the problem of finding an optimal switching surface. This switching surface $\sigma(x) = 0$ can be seen as dividing the state space into a positive and a negative side, $\sigma(x) \geq 0$ and $\sigma(x) < 0$ respectively. If the state is on the positive side of the surface, the maximum control value is used; if the state is on the negative side of the surface, then the minimum control value is used. If we restrict the admissible switching surfaces to be, say, hyperplanes in the state space $\mathbb{R}^n$, then they can be described by $n+1$ parameters. Therefore each component of the bang-bang feedback strategy can be parameterized by $n+1$ scalars, reducing significantly the complexity of the optimal feedback problem.

II. THE SYSTEM AND THE STRUCTURE OF THE FEEDBACK CONTROL

Our objective is to drive to a given target set Θ $(\subset \mathbb{R}^n)$ the state of the nonlinear system

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t), d(t)) \quad \text{a.e. } t \geq 0, \\ &x(0) = x_0 \in X_0, \\ &x(t) \in X \quad \text{for all } t \geq 0, \\ &u(t) \in U \quad \text{a.e. } t \geq 0, \\ &d(t) \in D \quad \text{a.e. } t \geq 0, \end{aligned} \tag{1}$$

where $X_0 \subset \mathbb{R}^n$ is the set of possible initial states, $X \subset \mathbb{R}^n$ is the set of possible states of the trajectory, $U \subset \mathbb{R}^m$ is the set of possible control values, $D \subset \mathbb{R}^p$ is the set of possible disturbance values, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is a given function. The state at time t from the trajectory x, starting from x_0 at t_0, and solving (1) is denoted $x(t; t_0, x_0, u, d)$ when we want to make explicit the dependence on the initial state, control and disturbance. It is also convenient to define, for some $T \geq t_0$, the function spaces

$$\mathcal{U}([t_0, T]) := \{u : [t_0, T] \to \mathbb{R}^m : u(t) \in U\}, \qquad \mathcal{D}([t_0, T]) := \{d : [t_0, T] \to \mathbb{R}^p : d(t) \in D\}.$$

Assumptions. We assume that U and D are boxed sets containing the origin, of the type

$$U = [u_1^{min}, u_1^{max}] \times [u_2^{min}, u_2^{max}] \times \ldots \times [u_m^{min}, u_m^{max}],$$
$$D = [d_1^{min}, d_1^{max}] \times [d_2^{min}, d_2^{max}] \times \ldots \times [d_p^{min}, d_p^{max}].$$

The target set Θ is a closed set, contains the origin and is robustly invariant under no control. That is, $x(t; 0, x_0, u, d) \in \Theta$ for all $t \in \mathbb{R}_+$, all $x_0 \in \Theta$, and all $d \in \mathcal{D}([0, t))$ when $u \equiv 0$. We further assume that f is a continuous function and locally Lipschitz continuous with respect to x. (This last requirement is necessary to guarantee uniqueness of trajectories. For stability results without such requirement see e.g. [20], [4].)

Since we allow discontinuous feedbacks some care is required to interpret the solution to the dynamic equation (1). This is because the solution to a differential equation with discontinuous right-hand side is not defined in a classical (Caratheodory) sense (see [21] for details). There are a few alternative definitions of solutions to ordinary differential with discontinuous right-hand side. The best known is the concept of Filippov solutions, which possesses some robustness and other desirable properties. However, it was shown in [22], [23] that there are controllable systems –the unicycle, for example– that cannot be stabilized, even allowing discontinuous feedbacks, if the trajectories are interpreted in a Filippov sense. Another way to define feedback strategies in differential games was recently proposed in [24] which is to interpret the discontinuous feedbacks laws as non-anticipative mappings between the control function and the disturbance.

A solution concept that has been proved successful in dealing with stabilization by discontinuous feedbacks is the concept of CLSS solution [25]. This solution concept was developed from works of Krasovskii and Subbotin in a context of differential games [26], [27], and has later been shown to combine successfully with stabilizing MPC approaches [12], [28]. It is, therefore, the concept used here. We describe it as follows.

Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ in $[0, +\infty)$ with $t_0 < t_1 < t_2 < \ldots$ and such that $t_i \to \infty$ as $i \to \infty$. Let the function $t \mapsto \lfloor t \rfloor_\pi$ give the last sampling instant before t, that is

$$\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}.$$

For such sequence π,

$$\dot{x}(t) = f(x(t), k(x(\lfloor t \rfloor_\pi))), \quad x(0) = x_0.$$

That is, the feedback is not a function of the state at every instant of time, rather it is a function of the state at the last sampling instant.

The MPC algorithm, described in the next section, implements naturally this solution concept. As a consequence, the resulting closed-loop trajectories are well-defined, even when discontinuous feedbacks are used.

III. The MPC strategy

Consider an auxiliary feedback law k^{aux}. Define a parameterization of a feedback law k^Λ such that defining the parameter matrix Λ defines the feedback $k^\Lambda(x)$ for all $x \in \mathbb{R}^n$. Moreover, the parameterization is done in such a way that when $\Lambda = 0$, we have $k^\Lambda = k^{aux}$. (A concrete example of one such parameterization is provided in a later section.) We shall call $\mathcal{K}$ to the space of all feedback laws obtained through this parameterization.

Consider a sequence of sampling instants $\pi := \{t_i\}_{i\geq 0}$ with constant inter-sampling times $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Let the control horizon T_c and prediction horizon T_p, with $T_c \leq T_p$, be multiples of δ ($T_c = N_c\delta$ and $T_p = N_p\delta$ with $N_c, N_p \in \mathbb{N}$). Consider also a terminal set S ($\subset \mathbb{R}^n$), a terminal cost function $W : \mathbb{R}^n \to \mathbb{R}$, and a running cost function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

The quantities time horizons T_c and T_p, objective functions L and W, terminal constraint set S, the inter-sampling time δ, and auxiliary feedback strategy k^{aux} are the quantities we are able to tune — the so-called design parameters— and should be chosen to satisfy the robust stability condition described in the next section.

At a certain instant $t \in \pi$, we select for the prediction model the control strategy for the intervals $[t, t + T_c)$ and $[t + T_c, t + T_p)$ in the following way. In the interval $[t, t + T_c)$, we should select, by solving an optimization problem, N_c matrices $\Lambda_1, \Lambda_2, \ldots, \Lambda_{N_c}$, defining this way the feedbacks k^{Λ_j} for $j = 1, 2, \ldots, N_c$. The strategy k^{aux}, known a priori, is used in the interval $[t+T_c, t+T_p)$. The optimization problem is a finite horizon differential game where the disturbance d acts as the maximizing player and the strategies $u = k^{\Lambda_j}$ act as the minimizing player.

The robust feedback MPC strategy is obtained by repeatedly solving on-line, at each sampling instant t_i, a min-max optimization problem $\mathcal{P}(x_{t_i}, T_c, T_p)$, to select the N_c matrices $\Lambda_1, \Lambda_2, \ldots, \Lambda_{N_c}$, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(x_t, T_c, T_p)$: $\mathrm{Min}_{\Lambda_1,\Lambda_2\ldots,\Lambda_{N_c}} \mathrm{Max}_{d\in\mathcal{D}([0,T_p])}$

$$\int_t^{t+T_p} L(x(s), u(s))ds + W(x(t + T_p)) \tag{2}$$

subject to:

$$x(t) = x_t$$

$$\dot{x}(s) = f(x(s), u(s), d(s)) \qquad \text{a.e. } s \in [t, t + T_p] \tag{3}$$

$$x(s) \in X \qquad \text{for all } s \in [t, t + T_p]$$

$$x(t + T_p) \in S. \tag{4}$$

where

$$u(s) = k^{\Lambda_j}(x(\lfloor s\rfloor_\pi)) \quad s \in [t + (j-1)\delta, t + j\delta), \quad j = 1, \ldots, N_c$$

$$u(s) = k^{aux}(x(\lfloor s\rfloor_\pi)) \quad s \in [t + (j-1)\delta, t + j\delta), \quad j \in N_c + 1, \ldots, N_p$$

In this optimization problem we use the convention that if some of the constraints is not satisfied, then the value of the game is $+\infty$. This ensures that when the value of the game is finite, the optimal control strategy guarantees the satisfaction of the constraints for all possible disturbance scenarios.

The MPC algorithm performs according to a Receding Horizon strategy, as follows:

1) Measure the current state of the plant x_{t_i}.
2) Compute the N_c matrices $\Lambda_1, \Lambda_2, \ldots, \Lambda_{N_c}$, defining the feedbacks k^{Λ_j}, $j = 1, \ldots, N_c$, solution to problem $\mathcal{P}(x_{t_i}, T_c, T_p)$.
3) Apply to the plant the control given by the feedback law $k^{\Lambda_1}(x_{t_i})$ in the interval $[t_i, t_i + \delta)$, (discard all the remaining data for $t \geq t_i + \delta$).
4) Repeat the procedure from (1.) for the next sampling instant t_{i+1}

We note that the startegy k^{aux} may never be actually applied to the plant. It is only applied if it coincides with the best option, i.e. if $\Lambda_1 = 0$ is in the optimal solution to problem $\mathcal{P}(x_{t_i}, T_c, T_p)$.

IV. ROBUST STABILITY ANALYSIS

The main stability result is provided in this section. It states that if the design parameters are chosen to satisfy the robust stability conditions RSC, then the MPC strategy ensures steering to a certain target set Θ. The following definition will be used

Definition 4.1 (Playable Set): The *playable set* $\Omega(T_c, T_p, S)$ is the set of all initial states x_0 for which using the inter-sampling time δ and the auxiliary strategy k^{aux} there exists some control strategy $k \in \mathcal{K}$ for $[0, T_p]$ with $k = k_{aux}$ for $[T_c, T_p]$ such that

$$x(T_p; 0, x_0, k, d) \in S \quad \text{for all } d \in \mathcal{D}([0, T_p]).$$

Consider the following Robust Stability Condition RSC

RSC The design parameters: time horizons T_c and T_p, objective functions L and W, terminal constraint set S, inter-sampling time δ, and auxiliary feedback strategy k^{aux} satisfy

- RSC1 The set S is closed, contains the origin, and is contained in X. Also $k^{aux}(x) \in U$ for all $x \in S$
- RSC2 The function L is continuous, $L(0, 0) = 0$, and for all $u \in U$ we have that $L(x, u) \geq M(x)$ for some continuous function $M : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $M(x) > 0$ for all $x \in \mathbb{R}^n \setminus \Theta$, and $M(x) \to \infty$ as $\|x\| \to \infty$.
- RSC3 The function W is Lipschitz continuous and $W(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
- RSC4 The set of initial states X_0 is contained in the playable set $\Omega(T_c, T_p, S)$.
- RSC5 For each sampling instant $t \in \pi$ and each $x_t \in S \setminus \Theta$, and for all possible disturbances $d \in \mathcal{D}([t, t + \delta)])$ we have

$$W(x(t + \delta)) - W(x_t) \leq -\int_t^{t+\delta} L(x(s), k^{aux}(x_t))\, \mathrm{d}s \qquad (RSC5a)$$

$$x(t + \delta; t, x_t, k^{aux}, d) \in S \cup \Theta. \qquad (RSC5b)$$

Remark 4.1: Condition (RSC5b) requires the set S to be invariant under the control k^{aux}. Condition (RSC5a) is similar to the infinitesimal decrease condition of control Lyapunov functions (CLF). The main difference, and a significant one, is that it just has to be satisfied within S, which is much easier if S is conveniently chosen. Therefore, we do not need to know a global CLF for the system, which might be hard to find, and would define us immediately a stabilizing feedback law that we are seeking. The auxiliary feedback law k^{aux} just has to be stabilizable within S. However, if in addition k_{aux} can drive the system to S in time T_c, then choosing all matrices $\Lambda_1, \ldots, \Lambda_{N_c}$ equal to zero is an admissible solution to the optimization problem. Therefore, the MPC strategy can only perform better than the known control law k^{aux}.

The use of a nonsmooth W is necessary for generic choices of the terminal set S (for example, if $S = \mathbb{R}^n$). This is because there are some systems that do not admit a smooth CLF. The unicycle system studied here is precisely one of such systems. It has been shown [29], that the nonholonomic integrator does not admit a smooth CLF, and the unicycle system can be transformed into the nonholonomic integrator by an appropriate change of coordinates [30]. A locally Lipschitz CLF, on the other hand, is guaranteed to exist for every globally asymptotically controllable system, as shown by Rifford [31].

We are in the conditions to state the following stability result where the function $x \mapsto d_A(x)$ denotes the distance from a point x to the set A (i.e. $d_A(x) := \min_{y \in A} \|x - y\|$).

Theorem 4.2: Assume condition RSC is satisfied and that the differential games $\mathcal{P}(x_t, T_c, T_p)$ have a value for all $x_t \in X$. Then, for a sufficiently small inter-sampling time δ, the state approaches asymptotically the target set Θ, that is $d_\Theta(x(t)) \to 0$ as $t \to \infty$.

Remark 4.3: This notion of stability includes the usual attractiveness but not the Lyapunov stability concept. The use of this notion is justified by the fact that it is not possible to satisfy attractiveness and Lyapunov stability simultaneously for some systems that we would intuitively classify as controllable, such as a car-like vehicle model (see [28] for a further discussion).

Proof: At a certain sampling instant t_i, we measure the current state of the plant x_{t_i} and we solve problem $\mathcal{P}(x_{t_i}, T_c, T_p)$ obtaining as solution the feedback strategy $\bar{k}$ to which corresponds, in the worst disturbance scenario, the trajectory $\bar{x}$ and control $\bar{u}$. The value of differential game $\mathcal{P}(x_{t_i}, T_c, T_p)$ is given by

$$V_{t_i}(t_i, x_{t_i}) = \int_{t_i}^{t_i+T_p} L(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t_i + T_p)). \tag{5}$$

Consider now the family of problems $\mathcal{P}(x_t, T_c - (t - t_i), T_p - (t - t_i))$ for $t \in [t_i, t_i + \delta)$. These problems start at different instants t, but all terminate at the same instant $t_i + T_p$. Therefore in the worst disturbance scenario, by Bellman's principle of optimality we have that

$$V_{t_i}(t, \bar{x}(t)) = \int_{t}^{t_i+T_p} L(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t_i + T_p)). \tag{6}$$

Suppose that the worst disturbance scenario did not occur and so, at time t, we are at state $x^*(t)$ which is, in general, distinct from $\bar{x}(t)$. Because such scenario is more favorable, and by the assumption on the existence of value to the differential game we have that

$$V_{t_i}(t, x^*(t)) \leq V_{t_i}(t, \bar{x}(t)) \qquad \text{for all } t \in [t_i, t_i + \delta). \tag{7}$$

We may remove the subscript t_i from the value function if we always choose the subscript t_i to be the sampling instant immediately before t, that is (recall that $\lfloor t \rfloor_\pi = \max_i \{t_i \in \pi : t_i \leq t\}$)

$$V(t, x) := V_{\lfloor t \rfloor_\pi}(t, x).$$

Let $\hat{x}$ be the concatenation of predicted trajectories $\bar{x}$ for each optimization problem. That is for $i \geq 0$

$$\hat{x}(t) = \bar{x}^i(t) \quad \text{for all } t \in [t_i, t_i + \delta)$$

where $\bar{x}^i$ is the trajectory of a solution to problem $\mathcal{P}(x_{t_i}, T_c, T_p)$. Note that $\hat{x}$ coincides with x^* at all sampling instants $t_i \in \pi$.

The following lemma establishes a monotone decreasing property of V (see [32] for the proof).

Lemma 4.4: There exists an inter-sample time $\delta > 0$ small enough such that for any $t' < t''$

$$V(t'', x^*(t'')) - V(t', x^*(t')) \leq -\int_{t'}^{t''} M(\hat{x}(s))ds.$$

We can then write that for any $t \geq t_0$

$$0 \leq V(t, x^*(t)) \leq V(t_0, x^*(t_0)) - \int_{t_0}^{t} M(\hat{x}(s))ds.$$

Since $V(t_0, x^*(t_0))$ is finite, we conclude that the function $t \mapsto V(t, x^*(t))$ is bounded and then that $t \mapsto \int_{t_0}^{t} M(\hat{x}(s))ds$ is also bounded. Therefore $t \mapsto \hat{x}(t)$ is bounded and, since f is continuous and takes values on bounded sets of (x, u, d), $t \mapsto \dot{\hat{x}}$ is also bounded. Using the fact that x^* is absolutely continuous and coincides with $\hat{x}$ at all sampling instants, we may deduce that $t \mapsto \dot{x}^*(t)$ and $t \mapsto x^*(t)$ are also bounded. We are in the conditions to apply the following lemma, a modification of Barbalat's lemma, yielding the assertion in the theorem (the proof of the lemma can be found in [32]).

Lemma 4.5: Let A be subset of $\mathbb{R}^n$ containing the origin, and M be a continuous function such that $M(x) > 0$ for all $x \notin A$ and $M(x) = 0$ for some $x \in A$. Let $d_A(x)$ be the distance function from a point $x \in \mathbb{R}^n$ to the set A. Let also x^* be an absolutely continuous function on $\mathbb{R}$, and $\hat{x}$ a function coinciding with x^* at the points of a sequence $\pi = \{t_i\}_{i \geq 0}$, such that $\|\hat{x}(\cdot)\|_{L^\infty(0,\infty)} < \infty$, $\|\dot{\hat{x}}(\cdot)\|_{L^\infty(0,\infty)} < \infty$ $\|x^*(\cdot)\|_{L^\infty(0,\infty)} < \infty$, and $\|\dot{x}^*(\cdot)\|_{L^\infty(0,\infty)} < \infty$. If

$$\lim_{T \to \infty} \int_0^T M(\hat{x}(t))\, dt < \infty,$$

then

$$d_A(x^*(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

V. Parameterized Bang-Bang Feedback Laws

We describe here a possible parameterization of the feedback law. We are interested in feedback controls of bang-bang type. That is, for each state, the corresponding control must be at one of the extreme values of its range. The exception is the target set Θ where the control is chosen to be zero. The control will attain its maximum or minimum value depending on which side of a certain surface the state is. More precisely, for each control component $j = 1, \ldots, m$

$$k_j(x) = \begin{cases} 0 & \text{if } x \in \Theta, \\ u_j^{max} & \text{if } \sigma_j(x) \geq 0, \\ u_j^{min} & \text{if } \sigma_j(x) < 0, \end{cases} \tag{8}$$

The function σ_j is a component of the switching function σ, and is associated with the switching surface $\sigma_j(x) = 0$ which divides the state-space in two.

Since these surfaces must be parameterized in some way to be chosen in an optimization problem, we will define them to have a fixed part σ^{aux}, possibly nonlinear, and a variable part σ^Λ which is affine and defined by a parameter matrix Λ.

$$\sigma(x) = \sigma^{aux}(x) + \sigma^\Lambda(x). \tag{9}$$

For each component $j = 1, 2, \ldots, m$, we have that $\sigma_j^\Lambda = 0$ is the equation of an hyperplane which is defined by $n + 1$ parameters as

$$\sigma_j^\Lambda(x) := \lambda_{j,0} + \lambda_{j,1}\, x_1 + \ldots + \lambda_{j,n}\, x_n. \tag{10}$$

(Note: the half-spaces $\sigma_j^\Lambda(x) \geq 0$ and $\sigma_j^\Lambda(x) < 0$ are not affected by multiplying all parameters by a positive scalar, therefore we can fix one parameter, say $\lambda_{j,0}$, to be in $\{-1, 0, 1\}$.) In total, for all components, there will be $m \times (n + 1)$ parameters to choose from. Selecting the parameter matrix

$$\Lambda := \begin{bmatrix} \lambda_{1,0} \cdots \lambda_{1,n} \\ \cdots \\ \lambda_{m,0} \cdots \lambda_{m,n} \end{bmatrix},$$

we define the function

$$\sigma^\Lambda(x) = \Lambda \begin{bmatrix} 1 \\ x \end{bmatrix}, \tag{11}$$

and therefore we define the switching function σ by (9) and feedback law k^Λ by (8). Each component of the feedback law is then described as

$$k_j^\Lambda(x) = \begin{cases} 0 & \text{if } x \in \Theta, \\ u_j^{max} & \text{if } \left[\sigma^{aux}(x) + \Lambda \begin{bmatrix} 1 \\ x \end{bmatrix}\right]_j \geq 0, \\ u_j^{min} & \text{if } \left[\sigma^{aux}(x) + \Lambda \begin{bmatrix} 1 \\ x \end{bmatrix}\right]_j < 0. \end{cases}$$

VI. Example: A unicycle system

Consider a unicycle mobile robot described by the following model

$$\begin{cases} \dot{x}(t) = [1 + d(t)] \cdot [u_1(t) + u_2(t)] \cdot \cos\theta(t) \\ \dot{y}(t) = [1 + d(t)] \cdot [u_1(t) + u_2(t)] \cdot \sin\theta(t) \\ \dot{\theta} = [1 + d(t)] \cdot [u_1(t) - u_2(t)], \end{cases}$$

where $\theta(t) \in [-\pi, \pi]$; $u_1, u_2(t) \in [-1, 1]$; and $d(t) \in [-d_{max}, d_{max}]$. Assume that $0 < d_{max} \leq 1/4$, and let $X_0 = X = \{(x, y, \theta) : \|(x, y)\| \leq R\}$ for some $R > 0$.

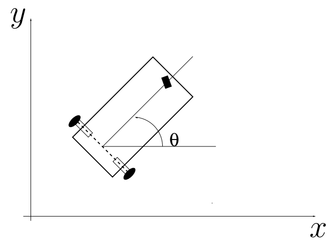

Fig. 1. A unicycle mobile robot.

The coordinates (x, y) are the position in the plane of the midpoint of the axle connecting the rear wheels, and θ denotes the heading angle measured anticlockwise from the x-axis. The controls u_1 and u_2 are the angular velocity of the right wheel and of the left wheel respectively. If the same velocity is applied to both wheels, the robot moves along a straight line (maximum forward velocity if $u_1 = u_2 = 1$). The robot can turn by choosing $u_1 \neq u_2$ (if $u_1 = -u_2 = 1$ the robot turns anticlockwise around the midpoint of the axle). The disturbance d is a multiplicative perturbation acting on the velocity of the wheels.

Our objective is to drive this system to the target set $\Theta = \{(x, y, \theta) : \|(x, y)\| \leq \epsilon_1, |\theta| \leq \epsilon_2\}$ for given $\epsilon_1, \epsilon_2 > 0$.

The MPC control law is obtained with the algorithm described in Section III with the following parameters.

Auxiliary control law. A possible stabilizing strategy, not necessarily the best, might be the following. (i) Choose a positive number ϵ_0 and reduce ϵ_1 if necessary so that $\epsilon_0 < \epsilon_1/2 < \epsilon_2/2$. (ii) Rotate the robot until its heading angle θ is directed to a point at a distance less than ϵ_0 from the origin of the plane. (iii) Move forward until reaching the origin of the plane or an ϵ_1 distance of it. (iv) Rotate again until θ is smaller that ϵ_2.

To formally describe this strategy it is convenient to define $\phi(x, y)$ to be the angle that points to the origin from position (x, y) away from the origin, more precisely

$$\phi(x,y) = \begin{cases} 0 & \text{if } \|(x,y)\| \leq \epsilon_1; \\ -(\pi/2)\,\text{sign}(y) & \text{if } \|(x,y)\| > \epsilon_1, x = 0, y \neq 0; \\ \tan^{-1}(y/x) + \pi & \text{if } \|(x,y)\| > \epsilon_1, x > 0; \\ \tan^{-1}(y/x) & \text{if } \|(x,y)\| > \epsilon_1, x < 0; \end{cases}$$

Note that $\phi(x, y)$ is conventionally defined to be equal to zero when $\|(x, y)\| \leq \epsilon_1$. Similarly, we define $\phi_1(x, y)$ and $\phi_2(x, y)$ to be the angles pointing from position (x, y) to a point in the x and y axis respectively, distancing ϵ_0 from the origin. (see fig. 2)

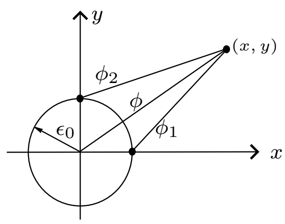

Fig. 2. Directions approaching points near to the origin.

$$\phi_1(x,y) = \begin{cases} 0 & \text{if } \|(x,y)\| \le \epsilon_1; \\ -(\pi/2)\,\mathrm{sign}(y) & \text{if } \|(x,y)\| > \epsilon_1, x-\epsilon_0 = 0 \text{ or } x+\epsilon_0 = 0; \\ \tan^{-1}(y/(x-\epsilon_0)) + \pi & \text{if } \|(x,y)\| > \epsilon_1, x \ge 0, x-\epsilon_0 \ne 0; \\ \tan^{-1}(y/(x+\epsilon_0)) & \text{if } \|(x,y)\| > \epsilon_1, x < 0, x+\epsilon_0 \ne 0; \end{cases}$$

$$\phi_2(x,y) = \begin{cases} 0 & \text{if } \|(x,y)\| \le \epsilon_1; \\ -(\pi/2)\,\mathrm{sign}(y) & \text{if } \|(x,y)\| > \epsilon_1, x = 0; \\ \tan^{-1}((y-\epsilon_0)/x) + \pi & \text{if } \|(x,y)\| > \epsilon_1, x > 0, y \ge 0; \\ \tan^{-1}((y+\epsilon_0)/x) + \pi & \text{if } \|(x,y)\| > \epsilon_1, x > 0, y < 0; \\ \tan^{-1}((y-\epsilon_0)/x) & \text{if } \|(x,y)\| > \epsilon_1, x < 0, y \ge 0; \\ \tan^{-1}((y+\epsilon_0)/x) & \text{if } \|(x,y)\| > \epsilon_1, x < 0, y < 0; \end{cases}$$

The feedback law k^{aux} is such that moves the robot forward if the heading angle θ is in between $\phi_m = \min\{\phi_1, \phi_2\}$ and $\phi_M = \max\{\phi_1, \phi_2\}$, and rotates it otherwise.

$$(u_1, u_2) = k^{aux}(x,y,\theta) = \begin{cases} (0,0) & \text{(Stop)} & \text{if } (x,y,\theta) \in \Theta; \\ (1,1) & \text{(Forward)} & \text{if } \phi_m \le \theta \le \phi_M \\ (1,-1) & \text{(Anticlockwise)} & \text{if } \theta < \phi_m \\ (-1,1) & \text{(Clockwise)} & \text{if } \theta > \phi_M \end{cases}$$

That is

$$u_1 = k_1^{aux}(x,y,\theta) = \begin{cases} 0 & \text{if } (x,y,\theta) \in \Theta; \\ 1 & \text{if } \theta \le \phi_M; \\ -1 & \text{if } \theta > \phi_M; \end{cases} \qquad u_2 = k_2^{aux}(x,y,\theta) = \begin{cases} 0 & \text{if } (x,y,\theta) \in \Theta; \\ 1 & \text{if } \theta \ge \phi_m; \\ -1 & \text{if } \theta < \phi_m; \end{cases}$$

So, the auxiliary switching function is

$$\sigma_1^{aux}(x,y,\theta) = -\theta + \phi_M(x,y), \qquad \sigma_2^{aux}(x,y,\theta) = \theta - \phi_m(x,y).$$

The MPC strategy will select at each sampling instant a matrix $\Lambda \in \mathbb{R}^{2\times 4}$ defining a feedback law k^Λ through the following switching function

$$\sigma(x,y,\theta) = \begin{bmatrix} \phi_M(x,y) - \theta \\ -\phi_m(x,y) + \theta \end{bmatrix} + \Lambda \begin{bmatrix} 1 & x & y & \theta \end{bmatrix}^T.$$

Terminal set. Define the terminal set S to be the set of states heading towards an ϵ_0-ball around the origin of the plane together with the target set and the origin of the plane, that is

$$S := \{(x,y,\theta) \in \mathbb{R}^2 \times [-\pi,\pi] : \phi_m(x,y) \le \theta \le \phi_M(x,y) \vee (x,y,\theta) \in \Theta \vee (x,y) = (0,0)\}.$$

Prediction and control horizons. The prediction horizon is chosen longer than the maximum time necessary to steer any state to the set S, that is the time to complete an 180 degrees turn with the worst possible disturbance

$$T_p = \frac{2\pi}{3} \ge \frac{\pi}{2(1-d_{max})}.$$

The control horizon T_c does not affect robust stability; it can be any number between δ and T_p. Then, the choice of the control horizon must consider a trade-off between performance and computational burden. Obviously, because $\Lambda_i \equiv 0$ is an admissible solution to the optimization problem, the MPC controller, based on solving optimization problems, performs better than the auxiliary strategy with respect to the considered objective function. For a deeper discussion on the use of two different control and prediction horizons see [33].

Objective functions L and W. Define the running cost and the terminal cost functions simply as

$$L(x, y, \theta) = x^2 + y^2 + \theta^2, \qquad W(x, y, \theta) = 2\int_0^{\bar{t}} L(x(t), y(t), \theta(t))\, dt,$$

where $\bar{t}$ is the time to reach the origin in the nominal case (i.e. with $d = 0$) and the strategy k^{aux} with $\epsilon_0 = 0$. An explicit formula is (see [32]) $W(x, y, \theta) = \frac{1}{3}(r^3 + |\theta|^3) + r\theta^2$ with $r = \sqrt{x^2 + y^2}$.

Inter-sampling time To satisfy RSC we should choose $\delta > 0$ such that

$$\delta \leq \min \left\{ \epsilon_1, \epsilon_2, \frac{\sin^{-1}(\epsilon_0/R)}{2(1 + d_{max})} \right\}.$$

The inequality with the last expression is required when we are far from the origin. In such situation, the angle $|\phi_1 - \phi_2|$ becomes small. We must therefore guarantee that when we are outside S and start rotating towards S during δ seconds, the robot would not cross to the other side of the cone S.

A detailed verification that the parameters introduced above fulfill the condition RSC — which ensures steering to the target set Θ — can be found in [32]. Conditions RSC1 to RSC4 and RSCb are directly verifiable in an easy way. To verify condition RSC5a, it is convenient to analyse separately the cases when $\|(x, y)\| \leq \epsilon_1$ — in which we use the control $u_1 = -u_2 = -\text{sign}(\theta)$; and (ii) when $\|(x, y)\| > \epsilon_1$ — in which we use the control $u_1 = u_2 = 1$.

VII. Conclusions

In this paper we address the problem of robust stabilizing constrained nonlinear systems using discontinuous state-feedback control laws. The control laws obtained are of a bang-bang type and are derived using a MPC technique based on the solution of a finite-horizon min-max optimization problem with respect to closed-loop strategies. Conditions under which steering to a set is guaranteed are established. A set of parameters satisfying all these conditions for the control of a unicycle mobile robot are derived. Three features used to reduce the computational burden are noteworthy: i) the use of discontinuous control strategies; ii) the use of bang-bang control law described with the switching surfaces parameterized with a possible small number of parameters; iii) the use of two different prediction and control horizons.

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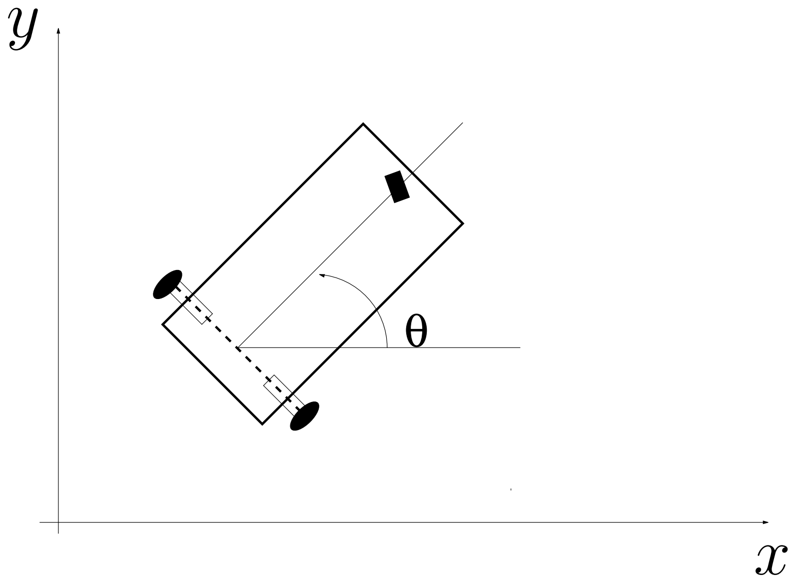
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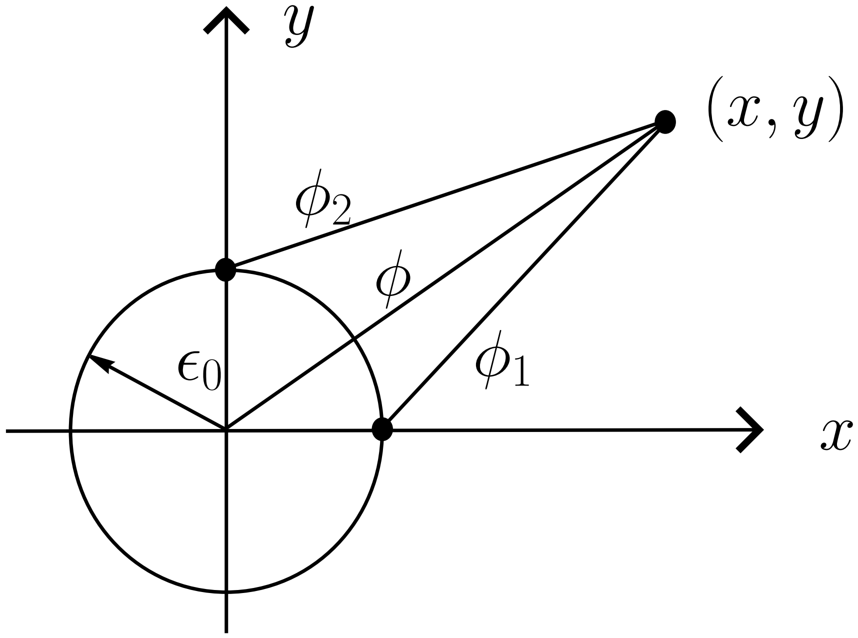
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