A General Framework to Design Stabilizing Nonlinear Model Predictive Controllers

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Abstract

This paper proposes a new Model Predictive Control (MPC) framework to generate feedback controls for time-varying nonlinear systems with input constraints. It provides a set of conditions on the design parameters that permits to verify *a priori* the stabilizing properties of the control strategies considered. The supplied sufficient conditions for stability can also be used to analyse the stability of most previous MPC schemes. The class of nonlinear systems addressed is significantly enlarged by removing the traditional assumptions on the continuity of the optimal controls and on the stabilizability of the linearized system. Some important classes of nonlinear systems, including some nonholonomic systems, can now be stabilized by MPC. In addition, we can exploit increased flexibility in the choice of design parameters to reduce the constraints of the optimal control problem, and thereby reduce the computational effort in the optimization algorithms used to implement MPC.

Key words: Predictive control; receding horizon; stabilizing design parameters; nonlinear stability analysis; discontinuous feedback; optimal control.

1 Introduction

This work concerns the construction of stabilizing feedbacks laws for nonlinear systems by the Model Predictive Control (MPC) method, which is also known as Receding-Horizon or Moving-Horizon Control. This method obtains the feedback control by solving a sequence of open-loop optimal control problems, each of them using the measured state of the plant as its initial state.

The class of systems addressed comprises time-varying nonlinear systems with input constraints, without pathwise state constraints and for which the state is available for measurement. The focus of our analysis is the guarantee of nominal stability of the system resulting from applying the MPC strategy.

The study of MPC stabilizing schemes has been the subject of intense research in recent years. The first stability results for nonlinear systems required the optimal control problems to constrain the terminal state to be at the origin [14,17]. These works were succeeded by other important contributions like the dual-mode approach [20], the contractive constraints [27], and more recently approaches based on the use of an appropriate terminal cost in the open-loop optimal control problems [5,8,13]. The importance of the

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terminal cost to guarantee stability was first noticed in [23] in a context of linear systems. The use of a terminal cost has been having its importance recognized in the last years and it is also of key importance in our approach. Further details and references can be found in the recent surveys on nonlinear model predictive control schemes focusing on stability: [16], [18] and [4].

Robustness and performance issues are not discussed here. In some cases, however, stability margins and optimality with respect to a modified problem could be established for MPC (see e.g. [15]).

Traditionally, MPC schemes with guaranteed stability for nonlinear systems impose conditions on the open-loop optimal control problem that either lead to some demanding hypotheses on the system or make the on-line computation of the open loop optimal control very hard. In previous work, these conditions take the form of a terminal state constrained to the origin, or an infinite horizon, or else impose some rather conservative controllability conditions on the system near the origin. (We show that none of the cited schemes is guaranteed to stabilize the trivial system $\dot{x} = x \cdot u$, with $\|u\| \leq 1$!) These schemes restrict the applicability of the MPC method, not only by narrowing the classes of systems to which it can be applied, but also by making it very difficult to verify whether some hypotheses are satisfied for a particular nonlinear system.

Most practitioners of MPC methods know that for some systems, by an appropriate choice of some parameters of the objective function and horizon (obtained by trial-and-error and some empirical rules), it is possible to obtain stabilizing trajectories without imposing demanding artificial constraints. However, their achievements cannot often be supported by any theoretical result to date. Also, "playing" with the design parameters and test the result with simulations is an option frequently criticized by researchers (see e.g. [2]). Here we intend to reduce this gap between theory and practice.

We propose a very general framework of MPC for systems satisfying very mild hypotheses. The *design parameters* of the MPC strategy are chosen in order to satisfy a certain (sufficient) *stability condition*. Then, the closed-loop system resulting from applying MPC will have the desirable stability properties guaranteed.

From a theoretical point of view, we provide a unifying framework for stabilizing MPC schemes. Most MPC schemes can be constructed from our framework and their stability properties deducted from our stability results. But perhaps more important is that with the insight obtained by using the general framework we are able to construct MPC schemes capable of dealing with new classes of nonlinear systems. These include systems for which the linearization is not stabilizable and nonholonomic systems. We provide examples of these systems for which none of the cited MPC schemes is able to guarantee stability.

From a practical point of view, we give a stability condition that can be verified *a priori* (i.e. one not requiring trial-and-error simulations) to guarantee that a particular set of design parameters will lead to stability. Some examples are investigated to show how to choose the design parameters in order to satisfy the stability condition. The generality of the framework gives us an increased flexibility to choose a set of stabilizing design parameters. This flexibility can be explored to reduce the terminal constraints that are traditionally imposed on the optimal control problems (OCPs). This results in OCPs that can be solved quicker by current optimization algorithms.

Another contribution of this work is to relax a common assumption of all previous continuous-time MPC schemes: the continuity of the controls solving the open-loop optimal control problems as well as the continuity of the resulting feedback laws. This assumption, in addition to being very difficult to verify, was a major obstacle in enabling MPC to address a broader class of nonlinear systems. This is because some nonlinear systems cannot be stabilized by a continuous feedback as was first noticed in [25] and [3]. Among such systems are the nonholonomic systems, which frequently appear in practice.

However, if we allow discontinuous feedbacks, it would not be clear what should be the solution (in a classical sense) of the dynamic differential equation. Consider a time-varying feedback control $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. The classical definition of a trajectory of the system

$$\dot{x}(t) = f(t, x(t), k(t, x(t))) \quad t \in \mathbb{R},$$
$$x(t_0) = x_0,$$

depends on certain properties of the function f, as well as on the requirement that the feedback $x \mapsto k(t, x)$ is continuous.

This motivated the development of new concepts of solution to differential equations under a discontinuous feedback. Previous attempts to deal with discontinuous controls in a MPC context are [21] using Filippov solutions and approaches avoiding the continuity problem by considering a discrete-time framework (e.g. [19]). Later, Ryan [24] and Coron and Rosier [7] have shown that Filippov solutions cannot lead to stability results for general nonlinear systems. A successful approach that deals with discontinuous feedbacks to stabilize general nonlinear systems is to use the "sampling-feedbacks" of Clarke *et al.* [6]. In their definition of trajectory, the feedback is not a function of the state on *every* instant of time, rather it is a function of the state at the last sampling instant. But this, as we will see, coincides with the trajectories defined by our MPC framework where a positive inter-sampling time δ is considered. This allows our MPC framework to overcome the inherent difficulty of defining solutions of differential equations with discontinuous feedbacks. In this way, the class of nonlinear systems addressed can be broadened to include, for example, important instances of nonholonomic systems. Further developments of the results given here provide a constructor of stabilizing feedbacks for a large class of nonlinear systems [10].

2 The Model Predictive Control Framework

We shall consider a nonlinear plant with input constraints, where the evolution of the state after time t is predicted by the following model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t, \tag{1a}$$
$$x(t) = x_t, \tag{1b}$$
$$u(s) \in U(s). \tag{1c}$$

The data of this model comprise a set $X_0 \subset \mathbb{R}^n$ containing all possible initial states at the initial time t_0, a vector x_t (with $t \geq t_0$) that is the state of the plant measured at time t, a given function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a multifunction $U : \mathbb{R} \rightrightarrows \mathbb{R}^m$ of possible sets of control values. These data combined with a particular control function $u : [t, +\infty) \to \mathbb{R}^m$ define a trajectory $x : [t, +\infty) \to \mathbb{R}^n$.

We assume this system to be asymptotically controllable on X_0.

Our objective is to obtain a feedback law that (asymptotically) drives the state of our plant to the origin. This task is accomplished by using a MPC strategy. Consider a sequence of sampling instants $\{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(t, x_t, T)$: Minimize

$$\int_t^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T)), \tag{2}$$

subject to:

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \in [t, t+T], \tag{3}$$
$$x(t) = x_t,$$
$$u(s) \in U(s) \quad \text{a.e. } s \in [t, t+T],$$
$$x(t+T) \in S. \tag{4}$$

The domain of this optimization problem is the set of admissible processes, namely pairs (x, u) comprising a measurable control function u and the corresponding absolutely continuous state trajectory x which satisfy the constraints of $\mathcal{P}(t, x_t, T)$. The problem is said to be feasible if there exists at least an admissible process. A process $(\bar{x}, \bar{u})$ is said to solve $\mathcal{P}(t, x_t, T)$ if it globally minimizes (2) among all admissible processes. (A thorough discussion of optimal control problems can be found in [26].)

The notation adopted here is as follows. The variable t represents real time while we reserve s to denote the time variable used in the prediction model. The vector x_t denotes the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The trajectory is sometimes denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit the dependence on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes our optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

The MPC conceptual algorithm consists of performing the following steps at a certain instant t_i (see Fig. 1).

(1) Measure the current state of the plant x_{t_i}.
(2) Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.
(3) The control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ is applied to the plant, (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).

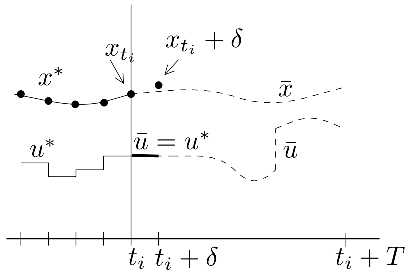

Fig. 1. The MPC strategy.

(4) The procedure is repeated from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law is a feedback control since during each sampling interval, the control u^* is dependent on the state x_{t_i}.

It is a well-known fact that (for fixed finite horizon) the closed-loop trajectory of the system (x^*) does not necessarily coincide with the open-loop trajectory ($\bar{x}$) solution to the OCP. Hence, the fact that MPC will lead to a stabilizing closed-loop system is not guaranteed *a priori*, and is highly dependent on the *design parameters* of the MPC strategy.

We show that we can guarantee stability of the resultant closed loop system, by choosing the design parameters to satisfy a certain *stability condition.* We anticipate here some of the key steps in our result. A crucial element of the stability condition is the requirement that the design parameters are chosen in such away that for all states x belonging to the set of possible terminal states of the OCP (which is, of course, a subset of S), there exists a control value $\tilde{u}$ such that

$$W_x(x) \cdot f(t, x, \tilde{u}) \;\leq\; -L(t, x, \tilde{u}). \tag{5}$$

This condition is important in establishing that a certain function V^δ constructed from value functions of the OCPs involved is "decreasing". Then, using Lyapunov-type arguments, we are able to prove the stability of the closed-loop system.

It is interesting to note that there were further recent developments in this direction. Mayne *et al.* [18] identified a condition similar to (5) for discrete time systems as a "common ingredient" to most stabilizing MPC schemes. Jadbabaie *et al.* [13] identified that if W is a control Lyapunov function (hence satisfying a condition like (5) for $S = \mathbb{R}^n$) then stability can be achieved. (The connection between MPC and control Lyapunov functions is also explored in Primbs *et al.* [22].) Here, we also show how stability of other MPC schemes can be verified with the help of (5). But a perhaps more important consequence is that this generalization enables us to construct MPC schemes guaranteeing stability for new important classes of nonlinear systems, like for example nonholonomic systems.

In the next section, we provide stability results for systems complying with the following hypotheses.

H1 For all $t \in \mathbb{R}^n$ the set $U(t)$ contains the origin, and $f(t, 0, 0) = 0$.

H2 The function f is continuous, and $x \mapsto f(t, x, u)$ is locally Lipschitz continuous for every pair (t, u).

H3 The set $U(t)$ is compact for all t, and for every pair (t, x) the set $f(t, x, U(t))$ is convex.

H4 The function f is compact on compact sets of x, more precisely given any compact set $X \subset \mathbb{R}^n$, the set $\{\, \|f(t, x, u)\| : t \in \mathbb{R}, x \in X, u \in U(t) \,\}$ is compact.

Remark 1 *It should be noted that these hypotheses are expressed directly in terms of the data of the nonlinear model. In contrast, most previous MPC literature explicitly assume at this stage feasibility of the OCP, or continuity of the optimal controls, or some properties of the value function. Here, feasibility will be guaranteed later by an appropriate choice of the design parameters and regularity of the controls or of the value function will not be imposed.*

Hypothesis H1 should not be seen as a restrictive one, since most systems can be made to satisfy it after an appropriate change of coordinates.

The assumption H3 (together with H2) is necessary to guarantee the existence of solution to the OCP. Some variations are possible. For example, if we allow relaxed controls (see [28]), the convexity of the velocity set $f(t, x, U(t))$ is no longer required.

The condition in H4 is automatically satisfied for time-invariant systems since the image of compact sets under a continuous function is compact.

3 Main Results

The main stability result is provided in this section. It asserts that the feedback controller resulting from the application of the MPC strategy is a stabilizing controller, as long as the design parameters satisfy the stability condition below.

Consider the following stability condition SC:

SC For system (1) the design parameters: time horizon T, objective functions L and W, and terminal constraint set S, satisfy:

SC1 The set S is closed and contains the origin.

SC2 The function L is continuous, $L(\cdot, 0, 0) = 0$, and there is a continuous positive definite and radially unbounded function $M : \mathbb{R}^n \to \mathbb{R}_+$ such that $L(t, x, u) \geq M(x)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^m$. Moreover, the "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t, x, u),\ \ell \geq L(t, x, u),\ u \in U(t)\}$ is convex for all (t, x).

SC3 The function W is positive semi-definite and continuously differentiable.

SC4 The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing X_0 such that for each pair $(t_0, x_0) \in \mathbb{R} \times X$ there exists a control $u : [t_0, t_0 + T] \to \mathbb{R}^m$ satisfying

$$x(t_0 + T; t_0, x_0, u) \in S.$$

Also, for all control functions u in the conditions above

$$x(t; t_0, x_0, u) \in X \quad \text{for all } t \in [t_0, t_0 + T].$$

SC5 There exists a scalar $\epsilon > 0$ such that for each time $t \in [T, \infty)$ and each $x_t \in S$, we can choose a control function $\tilde{u} : [t, t + \epsilon] \to \mathbb{R}^m$, with $\tilde{u}(s) \in U(s)$ for all $s \in [t, t + \epsilon]$, satisfying

$$\begin{aligned} &W_t(t, x_t) + W_x(t, x_t) \cdot f(t, x_t, \tilde{u}(t)) \\ &\quad \leq -L(t, x_t, \tilde{u}(t)) \end{aligned} \qquad (SC5a)$$

and

$$x(t + r; t, x_t, \tilde{u}) \in S \qquad (SC5b)$$

for all $r \in [0, \epsilon]$.

We start by providing a useful intermediate result that ensures the existence of solutions to the optimal control problems involved in the MPC strategy.

Proposition 2 *Assume hypotheses H1–H4. Assume also that the design parameters satisfy SC. Then for any $(t_0, x_0) \in \mathbb{R} \times X_0$ the solution to the open loop optimal control problem $\mathcal{P}(t_0, x_0, T)$ exists.*

Consider the sequence of pairs $\{t_i, x_i\}$ such that x_i is the value at instant t_i of a trajectory solving $\mathcal{P}(t_{i-1}, x_{i-1}, T)$. Then the solution to $\mathcal{P}(t_i, x_i, T)$ for all $i \geq 1$ also exists.

Moreover, the trajectory x^* constructed by MPC has no finite escape times.

The main result on stability is the following.

Theorem 3 *Assume the system satisfies hypotheses H1–H4. Choose the design parameters to satisfy SC . Then, for a sufficiently small inter-sample time δ, the closed-loop system resulting from the application of the MPC strategy is asymptotically stable in the sense that $\|x^*(t)\| \to 0$ as $t \to \infty$.*

The set of conditions in SC can be seen as divided into two types: the first type consists of the conditions guaranteeing the existence of solutions to the OCPs; the other type comprises the conditions ensuring that the closed-loop trajectory is actually driven towards the origin. This second type of condition naturally has similarities with the conditions requiring W to be a control Lyapunov function. At first, it might appear that finding W satisfying (SC5a) together with all the other conditions is a task as hard as finding a control Lyapunov function. However, a distinguishing feature of our approach is that (SC5a) is only required to be satisfied on a subset S, which we have considerable freedom to choose. An appropriate choice of S, as is shown in the next section, makes it easier to choose the remaining design parameters to satisfy SC.

A question that naturally arises is the relation between the existence of design parameters satisfying SC and the controllability of the system on X_0. The controllability of the system is obviously necessary to achieve stability (we cannot have a stabilizing feedback if there is no open-loop control driving the state to the origin). Also, a choice of design parameters satisfying SC implies stability, as shown. Therefore, by combining the two implications above, controllability must be a necessary condition for the existence of design parameters satisfying SC. However, the important converse question of whether there exists a set of stabilizing design parameters for every controllable system is left open in this work.

Remark 4 *The requirement in SC1 that the set S is closed is necessary to guarantee the existence of a solution to the open-loop optimal control problem.*

The first part of condition SC2 and condition SC3 are trivially satisfied for the usual quadratic objective function $L(x, u) = x^T Q x + u^T R u$, with $Q > 0$ and $R \geq 0$ and $W(x) = x^T P x$, with $P \geq 0$. The second part of SC2 on the convexity of the "extended velocity set" is a well known requirement for existence of solution in OCP with integral cost term. Given H3, it is automatically satisfied if L is convex and f depends linearly on u, or if L does not depend on u. The latter is a consequence of both $f(t, x, U(t))$ and $\{\ell : \ell \geq L(t, x)\}$ being convex sets.

A condition like SC4 is inevitable to guarantee the existence of an admissible process to the sequence of OCPs. We recall that in the hypotheses H1–H4 no assumption on the feasibility of the OCP was made. These hypotheses can be used in an initial test on the adequacy of the method since they involve only the data of the system model, which, typically, the designer is given or has little freedom to change. At a later stage, the designer can try to meet the desired *feasibility and stability properties by an appropriate choice of the design parameters.*

Condition (SC5a) is a key requirement for establishing the existence of a "decreasing" Lyapunov-like function, and thus asymptotic stability. It can be interpreted as the existence of a control $\tilde{u}$ that drives the state towards inner level sets of W at rate L. In the case of quadratic functions W the level sets are ellipsoids centred at the origin and the velocity vector should point inwards. However, W need not to be restricted to quadratic functions, and this freedom can be used in our advantage as we shall see below.

Condition (SC5b) simply states that the trajectory associated with $\tilde{u}$ does not leave the set S immediately.

The task of choosing design parameters to satisfy all the conditions of SC might seem formidable at first. But one should not be discouraged by the generality of SC. In fact, the stability condition greatly simplifies for some standard choices of part of the design parameters. Typically, we might choose the objective function to be quadratic (making SC2 and SC3 trivially satisfied); choose the set S to be the whole space $\mathbb{R}^n$ (makes SC4 trivially satisfied); or make SC5 trivially satisfied by choosing S to be the set of points that satisfy SC5. This issue is explored in the next section where we show, with the help of some examples, how stabilizing design parameters can be easily chosen.

4 Choosing Stabilizing Design Parameters

Here we analyse different strategies for choosing a set of design parameters satisfying the stability condition SC.

We start by the natural and easiest choice: setting the objective function to be quadratic and the terminal set as the whole space. This simplified framework works for linear systems and for some nonlinear systems as is shown in the first example below. A more thorough discussion of this choice is provided in [11,12]. There, we identify a class of nonlinear systems that can be stabilized with this simplified framework, and explore additional properties that can be established, such as a prescribed degree of exponential stability.

For more complex nonlinear systems it might be difficult to find design parameters satisfying SC with a large set as $S = \mathbb{R}^n$. This task, in particular choosing W to satisfy SC5, can be simplified if we restrict the set S to be just a subset of $\mathbb{R}^n$ containing the origin, for instance a linear subspace, a closed ball centred at the origin, or a set of trajectories approaching the origin. The last two examples describe this choice of S.

Starting by the first strategy described, we set S to be the whole space $S = \mathbb{R}^n$, and the objective function to be a quadratic positive definite function — where $L(x, u) = x^T Qx + u^T Ru$ (with $Q > 0$ and $R \geq 0$), and $W(x) = kx^T Px$ (with k a positive scalar and P a positive definite matrix). If f is affine in u or R is chosen to be zero then conditions SC2 and SC3 are satisfied. Since $S = \mathbb{R}^n$, conditions SC4 and (SC5b) are also trivially satisfied and SC reduces to the following simplified version of (SC5a).

SC′ The positive scalar k and the positive definite symmetric matrix P are such that for each $x \in \mathbb{R}^n$ belonging to the set of possible terminal states of the OCP, we can choose a control value $u \in U$ satisfying

$$2kx^T Pf(x, u) \leq -(x^T Qx + u^T Ru).$$

In this simplified framework, the stability result, Thm. 3 holds with SC replaced by SC′. The next example shows how to choose the design parameters satisfying SC′.

In all the examples that follow, the domain of attraction, the set of possible initial states X_0, can be set to any compact subset of $\mathbb{R}^n$.

Example 1: A simple nonlinear system Consider the nonlinear system with control constraints:

$$\dot{x}(t) = x(t) \cdot u(t), \qquad u(t) \in [-1, 1].$$

Although it is trivial to control this system, none of the previously cited MPC methods is able to guarantee its stability.

We can easily see that it is impossible to drive the state to the origin in a finite time, and that the linearization of the system is uncontrollable at the origin. Hence, trying to satisfy a terminal-state constraint, $S = \{0\}$, such as is required in the classical stability results for MPC [14,17] would fail. Also, the dual-mode, contractive constraint, or the quasi-infinite approach ([20], [27], [8] and [5]) do not guarantee stability since they require stabilizability of the linearized system at the origin.

Despite that, we can easily find design parameters such that SC′ is satisfied for this system. For example, if we set the design parameters

$$Q = I_n, \quad R = 0, \quad P = I_n, \text{ and } k = \frac{1}{2},$$

then, there exist a control $u = -1$ such that the stability condition SC′ is satisfied:

$$2kx^T P f(x,u) = -x^T P x = -\|x\|^2$$
$$= -x^T Q x.$$

Thus, this simple choice of design parameters guarantees closed-loop stability.

Example 2: A nonlinear system Consider the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + 2x_1(t)u(t), \end{cases}$$

with the control constraint

$$u(t) \in [0,1] \quad \text{a.e. } t.$$

This system cannot be driven to the origin in finite time; hence, MPC schemes having terminal state constrained to the origin do not apply. Moreover, linearizing around the origin we obtain the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x,$$

having poles over the imaginary axis at $\{-j, j\}$. Hence it is not stabilizable, and all the other cited MPC schemes fail to guarantee closed loop stability.

However, our framework enables us to almost trivially find the feedback control for this system. Simply notice that in the subspace

$$S := \{(x_1, x_2) : x_1 = -x_2\}$$

the control $u = 1$ drives the system towards the origin through S. Furthermore, this subspace can be reached with a well-determined finite horizon (see Fig. 2)

$$T = 2\pi,$$

because choosing $u = 0$, the trajectory is

$$\begin{cases} x_1(t) = x_1(0)\sin(t) \\ x_2(t) = x_2(0)\cos(t). \end{cases}$$

As to the objective function, the simple choice

$$L(x) = \|x\|^2 \text{ and } W(x) = \|x\|^2$$

is able to satisfy SC5 since

$$\dot{W} = 2x^T \cdot f(x,1) = -4x_1^2 \ \leq \ -L(x) = -2x_1^2,$$

and

$$x(t+r, t, x_t, 1) \in S \quad \text{for all } r \geq 0,$$

if $x_t \in S$.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

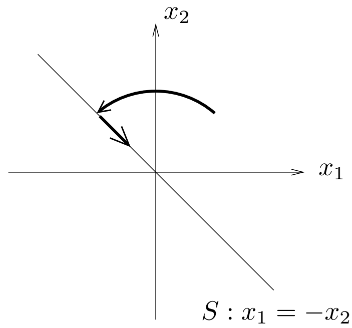

Fig. 2. Example 2: Reaching set S.

From this example we may conclude that using the terminal set to be a ball centred at the origin and linear feedback within it (as done in some MPC schemes) is clearly not the best choice for some systems, namely the important class of nonholonomic systems. This same conclusion can also be drawn from the next example.

Example 3: A car-like vehicle Consider the car-like vehicle of Fig. 3, steered by two front directional wheels, represented by the following model.

$$\begin{cases} \dot{x} = v \cdot \cos\theta \\ \dot{y} = v \cdot \sin\theta \\ \dot{\theta} = v \cdot c \end{cases}$$

where the control inputs v and c satisfy

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

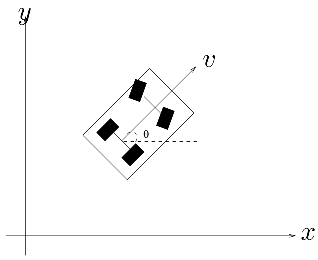

Fig. 3. A car-like vehicle.

Here (x, y) represents the location in the plane of a point in the car (the mid-point of the axle between the two rear wheels), and θ the angle of the car body with the x axis. The control v represents the linear velocity and c the curvature (the modulus of c is the inverse of the turning radius). It should be noted that the vehicle has a minimum turning radius ($R_{min} = |c_{max}|^{-1}$).

Our objective is to find a feedback controller to drive the vehicle to the origin ($x = y = 0$ and also $\theta = 0$). This objective cannot be achieved by any of the

MPC methods cited: firstly because the linearization of the system around the origin is not stabilizable; secondly because the system cannot be stabilized by a continuous feedback, since it is a nonholonomic system. (This last issue makes the stabilization of this system challenging; see [1].)

To define S we should look for possible trajectories approaching the origin. One possibility for such a set is represented in Fig. 4

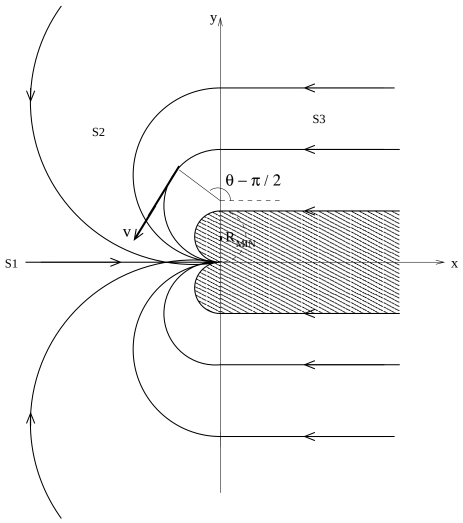

Fig. 4. Set S of trajectories approaching the origin.

We define S to be the union of all semi-circles with radius greater than or equal to R_{min}, with centre lying on the y axis, passing through the origin, and lying in the left half-plane. In order to make this set reachable in finite time from any point in the space, we add the set of trajectories that are horizontal lines of distance more than $2R_{min}$ from the x axis, and lie in the right half-plane. More precisely

$$S := S_1 \cup S_2 \cup S_3$$

where

$$S_1 := \{(x, 0, 0) : x \leq 0\}$$

$$S_2 := \Big\{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x \leq 0,\ x^2 + (y-r)^2 = r^2,\ r \geq R_{min},\ \tan\left(\theta - \frac{\pi}{2}\right) = \frac{y-r}{x}\Big\}$$

$$S_3 := \{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x > 0,\ |y| \geq 2R_{min},\ \theta = \pi\}$$

The set S as defined above can be reached in a well-determined finite time-horizon. This horizon is the time to complete a circle of minimum radius at maximum velocity, that is

$$T = 2\pi R_{min}/v_{max}.$$

Conditions SC1 and SC4 are satisfied.

Choose L as

$$L(x, y, \theta) := x^2 + y^2 + \theta^2,$$

and W as

$$W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t))dt$$

where $\bar{t}$ is the time to reach the origin with the controls chosen to be the maximum velocity and the curvatures depicted in Fig. 4. Of course, if we are already at the origin the value chosen for the velocity is zero. That is

$$v_{x,y,\theta} = \begin{cases} v_{max} & \text{if } (x, y, \theta) \neq 0 \\ 0 & \text{if } (x, y, \theta) = 0, \end{cases}$$

and

$$c_{x,y,\theta} = \begin{cases} 0 & \text{if } (x, y, \theta) \in S_1 \cup S_3 \\ -\text{sign}(\theta)/r & \text{if } (x, y, \theta) \in S_2 \cup \bar{S}, \end{cases}$$

where

$$r = \frac{x^2 + y^2}{2|y|}.$$

An explicit formula for W is

$$W(x, y, \theta) = \begin{cases} \dfrac{-x^3}{3v_{max}} \quad \text{if } (x, y, \theta) \in S_1 \\[2ex] \dfrac{r}{3v_{max}} [6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 \\ \quad +6r\,(x - \theta y) \cos(\theta) \\ \quad +6r\,(-r + \theta x + y) \sin(\theta)] \\ \quad \text{if } (x, y, \theta) \in S_2 \\[2ex] \dfrac{r}{3v_{max}} \left[x^2 + 3\pi^2 x + r\pi^3 + 30\pi r^2\right] \\ \quad \text{if } (x, y, \theta) \in S_3. \end{cases}$$

We can easily see that SC2 and SC3 are satisfied and a further calculation shows that SC5 also is fulfilled.

5 Comparison with Alternative MPC Approaches

In this section, we show how most of the previous approaches can be seen as particular cases of the general framework proposed, and emphasize the prominent role of the stability conditions, mainly SC5, in ensuring the stabilizing properties of each approach.

Terminal state constrained to the origin This approach is described in [14,17]. To cover this approach by our framework we would choose

$$L(x,u) = x^T Qx + u^T Ru,$$
$$W = 0,$$
$$S = \{0\},$$

The stabilizing properties of this approach can be confirmed by using our stability conditions. Condition SC5 is satisfied since $L(0,\cdot) = 0 = \dot{W}$, and so stability is guaranteed provided that S is reachable in time T. The main drawback of this approach is precisely the terminal state constraint, because the assumption on the existence of an admissible solution to the open loop optimal control problem is not always easy to verify. Another drawback is the difficulty of computing an exact solution to the constrained optimal control problem on-line.

Dual mode approach This approach was first described in [20]. In this case, outside a ball $S = \epsilon\mathbb{B}$ centred at the origin, we have to solve the open loop optimal control problem with

$$L(x,u) = x^T Qx + u^T Ru,$$
$$W = 0,$$
$$S = \epsilon\mathbb{B}$$
$$T \text{ free}$$

Before we reach S, we have a free time problem. After the set S is reached we switch to a linear stabilizing feedback controller for the linearized system.

Naturally, we would have to define S in such a way that the linear feedback controller stabilizes the actual system, which may not be easy if we have strong nonlinearities, or even impossible if the linearization of the system is not stabilizable.

The quasi-infinite approach discussed next is an evolution of this concept that does not require switching between controllers.

Terminal-Cost Based Approaches The quasi-infinite horizon approach of Chen and Allgöwer [5], as well as the approach of De Nicolao *et al.* [8], also use a terminal cost in the objective function as a key element to achieve stability. Their approaches have a very intuitive explanation that might also help to give some insight in our analytical results. The central idea is to choose the terminal cost such that it exceeds the running cost till infinity (using some stabilizing control after $t+T$)

$$W(x(t+T)) \geq \int_{t+T}^{\infty} L(x(s), u(s))\, ds. \tag{6}$$

This would imply that the value function for this problem is greater than the value function for the infinite horizon problem. Thus, using the value function as a Lyapunov function, stability can be easily proved provided that the control is stabilizing and, thereby, W is bounded. As determining W, and a stabilizing control, satisfying the above inequality in general might be difficult, W is computed in some ball centred at the origin (which we choose to be the terminal constraint set S), on which the controller is a stabilizing linear feedback for the linearization of the system around the origin.

In [8], a discrete-time analysis for time-varying systems is carried out. The function W is defined by (6) when equality is used, and is possibly a non-quadratic function. In contrast, [5] do a continuous-time analysis and W is a quadratic function – the Lyapunov function of the linearized system – which is easier to obtain.

Condition (6) is closely related to our condition in (SC5a). Let $(\tilde{x}, \tilde{u})$ be a stabilizing process defined in $[t_f, \infty)$, and

$$W(x(t_f)) = \int_{t_f}^{\infty} L(\tilde{x}(s), \tilde{u}(s))\, ds.$$

Differentiating with respect to time, we obtain

$$\dot{W}(x(t_f)) = -L(\tilde{x}(t_f), \tilde{u}(t_f)),$$

which yields our condition (SC5a).

The main disadvantage in these approaches is that they cannot be used if the linearization of the system around the origin is not stabilizable. Our framework can be seen as a generalization of the these approaches, but where the difficulties mentioned are overcomed by considering a more general terminal set S.

Global CLF based approaches Recently [22] and [13] have developed approaches that guarantee stability based on a global control Lyapunov function (CLF). Although their approaches highlight some interesting theoretical connections, they require that the inputs are unconstrained and that a global CLF for the system is known. In practice, MPC methods are most useful when the inputs are constrained [18], and when there is no alternative way to find a stabilizing feedback. If a global CLF is known we can immediately obtain a stabilizing feedback without having to solve the on-line optimization problems required by the MPC strategy. The use of MPC is justified in these cases, however, by performance issues.

Nevertheless, it is interesting to note that setting the set S to be the whole space and the functions W, L to be a control Lyapunov pair, the stability condition SC is satisfied.

6 Proof of the Results

Proof of the existence result (Prop. 2). Consider first $\mathcal{P}(t_0, x_0, T)$. Noticing that f and the objective function are continuous, $x \mapsto f(t, x, u)$ is Lipschitz, U is compact and non-empty, the "extended velocity set" is convex, the terminal set S is closed and nonempty, and SC4 guarantees the existence of an admissible process, we are in a position to apply a well-known existence result on solution to OCPs (see e.g. [9]). The first assertion follows.

Assume that the solution to $\mathcal{P}(t_{i-1}, x_{i-1}, T)$ exists and that $x_{i-1} \in X$ (X defined as in SC4). Pick a pair (t_i, x_i) from the trajectory solving this latter problem. Then from SC4, we have that $x_i \in X$, and we also satisfy all the conditions for $\mathcal{P}(t_i, x_i, T)$ to have existence of solution guaranteed. The second assertion follows by induction.

It remains to prove the third assertion. Notice that implicit in the existence of solution, we have that if $\bar{x}$ is a trajectory from a solution to $\mathcal{P}(t_i, x_i, T)$ then

$$\|\bar{x}\|_{L^\infty[t_i, t_i+T]} < \infty \quad \text{for all } i \geq 0.$$

As the MPC trajectory x^* is constructed with the concatenation of solutions to a sequence of problems $\mathcal{P}(t_i, x_i, T)$ in the conditions of the second assertion, we deduce that for all $M_1 \geq t_0$ there exists $M_2 \in \mathbb{R}$ such that

$$\|x^*\|_{L^\infty[t_0, M_1]} < M_2,$$

as required.

Next, we shall prove that the closed loop system resulting from the MPC strategy is asymptotically stable.

Proof of the main stability result (Thm. 3). We show that the "MPC value function" $V^\delta(t, x)$ constructed from value functions of OCPs, satisfies a decrescence condition implying that the closed loop system is asymptotically stable as required.

Consider the sampling interval $[t_i, t_i + \delta)$. Let $(\bar{x}, \bar{u})$ be our solution to $\mathcal{P}(t_i, x_{t_i}, T)$. By definition of the MPC strategy we have that

$$u^*(t) = \bar{u}(t) \quad \text{for all } t \in [t_i, t_i + \delta).$$

Assuming the plant behaves as predicted by the model in this interval (we are considering just nominal stability, not robust stability) we have also that

$$x^*(t) = \bar{x}(t) \quad \text{for all } t \in [t_i, t_i + \delta). \tag{7}$$

For $t \in [t_i, t_i + \delta)$ define

$$V_{t_i}(t, x_t)$$

to be the value function for problem $\mathcal{P}(t, x_t, T - (t - t_i))$ (the usual OCP but where we shrink the horizon by $t - t_i$). We have that $V_{t_i}(t, x^*(t)) = V_{t_i}(t, \bar{x}(t))$ and by Bellman's Principle of Optimality a solution to $\mathcal{P}(t, \bar{x}(t), T - (t - t_i))$ coincides with the remaining trajectory of $(\bar{x}, \bar{u})$ (because for all $t \in [t_i, t_i + \delta)$ all these problems terminate at the same instant $t_i + T$), (see Fig. 5) therefore

$$\begin{aligned} &V_{t_i}(t, x^*(t)) \\ &\quad = \int_t^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds \\ &\qquad\quad + W(t_i + T, \bar{x}(t_i + T)) \\ &\quad = V_{t_i}(t_i, \bar{x}(t_i)) - \int_{t_i}^{t} L(s, \bar{x}(s), \bar{u}(s))ds. \end{aligned} \tag{8}$$

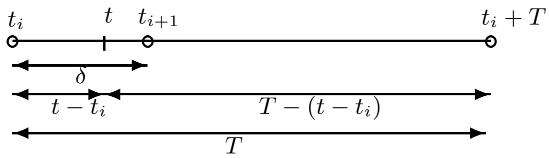

Fig. 5. Time intervals involved in problems $\mathcal{P}(t, x_t, T - (t - t_i))$.

Finally define the "MPC Value function" to be

$$V^\delta(t, x_t) := V_\tau(t, x_t)$$

where τ is the sampling instant immediately before t, that is $\tau = \max_i\{t_i : t_i \leq t\}$.

We show below that for the closed-loop trajectory x^* the function $t \mapsto V^\delta(t, x^*(t))$ converges to zero as $t \to \infty$ and thus x^* converges to zero as well. From (8) we know that this function is decreasing on each interval $(t_i, t_i + \delta)$ for any i. The next lemma establishes that V^δ is smaller at t_{i+1} than at t_i. (See Fig. 6).

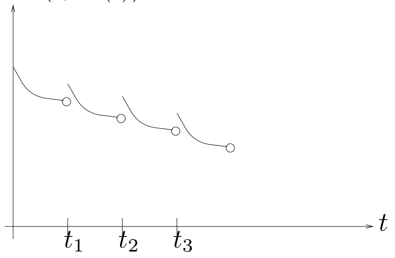

Fig. 6. "Decreasing" behaviour of the MPC value function $V^\delta(t, x^*(t))$.

Lemma 5 *There exists an inter-sample time $\delta > 0$ small enough such that*

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) \leq -\int_{t_i}^{t_{i+1}} M(x^*(s))ds \quad \textit{for all } i \in \mathbb{N}.$$

Proof. First notice that due to the assumption on the accuracy of the model (7)

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) = V_{t_{i+1}}(t_{i+1}, \bar{x}(t_{i+1})) - V_{t_i}(t_i, \bar{x}(t_i)).$$

The value function for $\mathcal{P}(t_i, x_{t_i}, T)$ is

$$V_{t_i}(t_i, \bar{x}(t_i)) = \int_{t_i}^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T, \bar{x}(t_i + T)).$$

Choose δ smaller than ϵ of SC5. Extend the process $(\bar{x}, \bar{u})$ to $[t_i, t_i + T + \delta])$ in such a way that $\bar{u} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^m$ satisfies SC5. To this control will correspond the extended trajectory $\bar{x} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^n$. The condition SC5 guarantees that the extended process $(\bar{x}, \bar{u})$ taken in the interval $[t_i+\delta, t_i+T + \delta]$ is admissible for problem $\mathcal{P}(t_i + \delta, x_{t_i+\delta}, T)$. But since this process is not necessarily optimal, we have

$$V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) \leq \int_{t_i+\delta}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T + \delta, \bar{x}(t_i + T + \delta)),$$

hence

$$\begin{aligned} &V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) \\ &\leq -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ &+W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) \\ &-W(t_i + T, \bar{x}(t_i + T)). \end{aligned}$$

Our choice of δ and (SC5b) imply

$$\bar{x}(t_i + T + r) \in S \quad \text{for all } r \in [0, \delta].$$

Integrating (SC5a) we have

$$W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) - W(t_i + T, \bar{x}(t_i + T)) + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \leq 0.$$

Finally, recalling the condition on M in SC2 we obtain

$$\begin{aligned} &V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) \\ &\leq -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ &\leq -\int_{t_i}^{t_i+\delta} M(x^*(s))ds. \end{aligned}$$

The lemma is proved. □

Lemma 6 *For all $t \geq t_0$*

$$V^\delta(t, x^*(t)) + \int_{t_0}^{t} M(x^*(s))ds \leq V^\delta(t_0, x^*(t_0)).$$

Proof. Let $t_i = i \cdot \delta$. (Without loss of generality we are assuming that $t_0 = 0$.) From Lemma 5 we easily obtain

$$V^\delta(t_i, x^*(t_i)) - V^\delta(0, x^*(0)) \leq -\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} M(x^*(s))ds$$

or

$$V^\delta(t_i, x^*(t_i)) \leq V^\delta(0, x^*(0)) - \int_0^{t_i} M(x^*(s))ds.$$

Using equality (8)

$$
\begin{aligned}
V^{\delta}(t, x^*(t)) &= V_{t_i}(t, x^*(t)) \\
&\leq V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^{t} M(x^*(s))ds \\
&\leq V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^{t} M(x^*(s)) \\
&\leq V^{\delta}(0, x^*(0)) - \int_{0}^{t} M(x^*(s))ds
\end{aligned}
$$

□

Now, from the last lemma, since M is positive definite, the function $t \mapsto V^{\delta}(t, x^*(t))$ is bounded for all $t \in [t_0, \infty)$. We may also deduce from the last lemma that $\int_{t_0}^{t} M(x(s))ds$ is bounded as well. We have that x^* is bounded and from the properties of f that $\dot{x}^*$ is also bounded. These facts combine with the following well known lemma (the proof of which can be found in e.g. [21]) to prove asymptotic convergence.

Lemma 7 *Let M be a continuous, positive definite function and x be an absolutely continuous function on* $\mathbb{R}$. *If*

$$
\begin{aligned}
&\|x(\cdot)\|_{L^\infty(0,\infty)} < \infty, \\
&\|\dot{x}(\cdot)\|_{L^\infty(0,\infty)} < \infty, \text{ and} \\
&\lim_{T\to\infty} \int_0^T M(x(t))\,dt < \infty,
\end{aligned}
$$

then

$$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$

The result of Thm. 3 follows immediately.

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