Nondegenerate Necessary Conditions for Nonconvex Optimal Control Problems with State Constraints

M. M. A. Ferreira[1,*] F. A. C. C. Fontes[2,†]

R. B. Vinter[2]

[1] Instituto de Sistemas e Robótica and

Departamento de Engenharia Electrotécnica e de Computadores

Faculdade de Engenharia da Universidade do Porto

4099 Porto Codex, Portugal

[2] Centre for Process Systems Engineering and

Department of Electrical and Electronic Engineering

Imperial College of Science, Technology and Medicine

London SW7 2BY, U.K.

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[†]On leave from Departamento de Matemática, and CMAT, Universidade do Minho, 4800 Guimarães, Portugal. Supported by grant BD 2918/94 from Programa PRAXIS XXI/FCT

Abstract

Standard versions of the maximum principle for optimal control problems with pathwise state inequality constraints are satisfied by a trivial set of multipliers in the case when the left endpoint is fixed and lies in the boundary of the state constraint set, and so give no useful information about optimal controls. Recent papers have addressed the problem of overcoming this degenerate feature of the necessary conditions. In these papers it is typically shown that, if a constraint qualification is imposed, requiring existence of inward pointing velocities, then sets of multipliers exist in addition to the trivial ones. A simple, new approach for deriving nondegenerate necessary conditions is presented, which permits relaxation of hypotheses previously imposed, concerning data regularity and convexity of the velocity set.

Key words. optimal control, maximum principle, state constraints, degeneracy

1 Introduction

Consider the optimal control problem

$$
\begin{array}{llll}
(P) & \text{Minimize} & g(x(1)) & (1)\\
 & \text{subject to} & & \\
 & & \dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. } t \in [0,1] & (2)\\
 & & x(0) = x_0 & \\
 & & x(1) \in C & \\
 & & u(t) \in \Omega(t) \quad \text{a.e. } t \in [0,1] & \\
 & & h\left(t, x(t)\right) \le 0 \quad \text{for all } t \in [0,1], & (3)
\end{array}
$$

for which the data comprises functions $g : \mathbb{R}^n \mapsto \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$, $h : [0,1] \times \mathbb{R}^n \mapsto \mathbb{R}$, and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$.

The set of *control functions* for (P) is

$$
\mathcal{U} := \left\{u : [0,1] \mapsto \mathbb{R}^m : u \text{ is a measurable function, } u(t) \in \Omega(t) \text{ a.e. } t \in [0,1]\right\}.
$$

The *state trajectory* is an absolutely continuous function which satisfies (2). The domain of the above optimization problem is the set of *admissible processes*, namely pairs (x, u) comprising a control function u and the corresponding state trajectory x which satisfy the constraints of (P). We say that an admissible process $(\bar{x}, \bar{u})$ is a *strong local minimizer* if there exists $\delta > 0$ such that

$$g(\bar{x}(1)) \leq g(x(1))$$

for all admissible processes (x, u) satisfying

$$\|x(t) - \bar{x}(t)\|_{L^\infty} \leq \delta.$$

Necessary conditions for such problems, in the form of a Maximum Principle, have been known for many years. (See [6], [8] and also references to the early Russian literature in [2]. An informal survey is provided by [5].) Early versions of the necessary conditions typically assert (under hypotheses the details of which do not concern us here) existence of an absolutely continuous function p, a non-negative regular Borel measure $\mu \in C^*([0,1], \mathbb{R})$, and a scalar $\lambda \geq 0$ satisfying

$$\mu\{[0,1]\} + ||p||_{L^\infty} + \lambda > 0,$$

$$-\dot{p}(t) = \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right) \cdot f_x\left(t, \bar{x}(t), \bar{u}(t)\right) \quad \text{a.e. } t \in [0,1],$$

$$-\left(p(1) + \int_{[0,1]} h_x(s, \bar{x}(s))\mu(ds) \right) \in N_C(\bar{x}(1)) + \lambda g_x(\bar{x}(1)),$$

$$\operatorname{supp}\{\mu\} \subset \left\{ t \in [0,1] : h\left(t, \bar{x}(t)\right) = 0 \right\},$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto \left(p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right) \cdot f\left(t, \bar{x}(t), u\right).$$

(The normal cone N_C is interpreted below.)

Now consider the case when

$$h(0, x_0) = 0. \tag{4}$$

(If we assume that $h_x(0, x_0) \neq 0$, this is the case when x_0 lies in the boundary of the state region.) It is easy to see that the above necessary conditions are satisfied at *any* feasible process $(\bar{x}, \bar{u})$ (local minimizer or not) for the choice of multipliers

$$\lambda = 0, \quad \mu = \delta_{\{0\}}, \quad p = -h_x(0, x_0). \tag{5}$$

Here $\delta_{\{0\}}$ denotes the unit measure concentrated at the left endpoint. In this case then no useful information is supplied about minimizers.

The case (4) is encountered in certain applications of interest (see [4], for discussion of this point) and there is a growing literature on refinements of earlier necessary conditions which assert existence of multiplier sets in addition to the trivial one (5), under a suitable constraint qualification. (See [1], [2] and [4].)

The constraint qualifications involved typically require

$$\inf_{u \in \Omega(t)} h_x(t, x_0) \cdot f(t, x_0, u) < 0, \tag{6}$$

for t near 0. Loosely speaking, this is the requirement that there exist control functions pushing the state away from the state constraint boundary.

A variety of nondegenerate necessary conditions have been derived, covering problems with nonsmooth as well as smooth data, problems in which the dynamic constraint involves a differential inclusion, or a differential equation, and in which the state constraint is formulated as a set inclusion as well as a functional inequality. (See [1], [2] and the references therein.) A feature of earlier work, treating nonsmooth data, is the need to impose hypotheses requiring

(a) the velocity set $f(t, x, \Omega(t))$ is convex,

(b) the data are Lipschitz continuous with respect to the time variable.

In [1], for example, these hypotheses have an important role in ensuring the closure of certain sets of functions and that certain perturbation terms introduced in the analysis can be suitably estimated.

Simple new methods are introduced in this paper for proving nondegenerate necessary conditions, based on applying standard necessary conditions to the optimal control problem (P), after an appropriate modification of the data "near" to the left endpoint has been made. Their main advantage is that they are valid even when hypotheses (a) and (b) above are violated. The price we pay for reducing the hypotheses in this way is that the constraint qualification (6) is replaced by

$$\inf_{u \in \Omega(t)} h_x(t, x_0) \cdot (f(t, x_0, u) - f(t, x_0, \bar{u}(t))) < 0$$

for t near 0. (Strictly speaking, we shall impose some "nonsmooth" uniform version of this hypothesis.) A similar hypothesis was imposed in [4]. This constraint qualification depends on strong local minimizer $(\bar{x}, \bar{u})$ and so is not, in general, directly verifiable. However in certain cases *a priori* regularity properties of optimal controls permit verification of this hypothesis (see [4]).

We conclude with some definitions. We define the limiting normal cone $N_C(\bar{x})$ to the closed set $C \in \mathbb{R}^k$ at $\bar{x} \subset C$ as

$$\begin{aligned} N_C(x) := \{\lim y_i : \text{there exist } x_i \xrightarrow{C} x,\ \{M_i\} \subset \mathbb{R}^+ \text{ s.t.} \\ y_i \cdot (z - x_i) \le M_i |z - x_i|^2 \text{ for all } z \in C\}. \end{aligned}$$

The limiting subdifferential of a lower semicontinuous function $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ is defined as

$$\partial f(x) := \{y : (y, -1) \in N_{\operatorname{epi} f}(x, f(x))\}.$$

We define also $\partial_x^> h(t, x)$ to be the following "hybrid" partial subdifferential of h in the x variable

$$\begin{aligned} \partial_x^> h(t, x) := \operatorname{co}\{\xi : \text{there exists } (t_i, x_i) \to (t, x) \text{ s.t.} \\ h(t_i, x_i) > 0\, \forall i,\ h(t_i, x_i) \to h(t, x),\ \text{and } \nabla_x h(t_i, x_i) \to \xi\}. \end{aligned}$$

Throughout $\mathbb{B}$ will denote the closed unit ball and $\operatorname{co} S$ the convex hull of a set S.

2 Main Results

There follows a "nondegenerate" version of the Maximum Principle for state constrained problems. For the strong local minimizer $(\bar{x}, \bar{u})$ of interest, the following hypotheses will be invoked. There exists a positive scalar δ' such that:

H1 The function $(t, u) \mapsto f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}$ measurable for each x. ($\mathcal{L} \times \mathcal{B}$ denotes the product σ-algebra generated by the Lebesgue subsets $\mathcal{L}$ of $[0, 1]$ and the Borel subsets of $\mathbb{R}^m$.)

H2 There exists a $\mathcal{L} \times \mathcal{B}$ measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|$$

for $x, x' \in \bar{x}(t) + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, 1]$. Furthermore there exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$\|f(t, x, u) - f(t, x', u)\| \leq K_f\|x - x'\|$$

for $x, x' \in \bar{x}(0) + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, \epsilon']$.

H3 The function g is Lipschitz continuous on $\bar{x}(1) + \delta'\mathbb{B}$.

H4 The endpoint constraint set C is closed.

H5 The graph of Ω is $\mathcal{L} \times \mathcal{B}$ measurable.

H6 The function h is upper semicontinuous and there exists a scalar $K_h > 0$ such that the function $x \mapsto h(t, x)$ is Lipschitz of rank K_h for all $t \in [0, 1]$.

Reference is also made to the following constraint qualification.

(CQ) (*constraint qualification*) If $h(0, x_0) = 0$ then there exist positive constants K_u, ϵ, ϵ_1, δ, and a control $\tilde{u} \in \mathcal{U}$ such that for a.e. $t \in [0, \epsilon)$

$$\|f(t, x_0, \bar{u}(t))\| \leq K_u, \quad \|f(t, x_0, \tilde{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta$$

for all $\zeta \in \partial_x^> h(s, x),\ s \in [0, \epsilon),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

Define the Hamiltonian

$$H(t, x, p, u) = p \cdot f(t, x, u).$$

Theorem 2.1 *Let $(\bar{x}, \bar{u})$ be a strong local minimizer for (P). Assume that (H1)–(H6) are satisfied. Assume also that the constraint qualification (CQ) is satisfied. Then there exists an absolutely continuous function $p : [0,1] \mapsto \mathbb{R}^n$, a measurable function γ, a nonnegative Radon measure $\mu \in C^*([0,1], \mathbb{R})$ and a scalar $\lambda \geq 0$ such that*

$$-\dot{p}(t) \in \operatorname{co} \partial_x H(t, \bar{x}(t), q(t), \bar{u}(t)) \quad a.e.\ t \in [0,1], \tag{7}$$

$$-q(1) \in N_C(\bar{x}(1)) + \lambda \partial g(\bar{x}(1)), \tag{8}$$

$$\gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-}a.e.\ , \tag{9}$$

$$\operatorname{supp}\{\mu\} \subset \{t \in [0,1] : h(t, \bar{x}(t)) = 0\}, \tag{10}$$

for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto H(t, \bar{x}(t), q(t), u) \tag{11}$$

and,

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda > 0, \tag{12}$$

where

$$q(t) = \begin{cases} p(t) + \int_{[0,t)} \gamma(s)\mu(ds) & t \in [0,1) \\ p(t) + \int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases}$$

In the above $\partial_x H$ denotes the limiting subdifferential with respect to the x variable.

Of course the main point of interest in these necessary conditions is the "non-triviality" condition (12) which replaces the traditional condition

$$\mu\{[0,1]\} + ||p||_{L^\infty} + \lambda > 0. \tag{13}$$

Notice that the degenerate set of multipliers

$$\lambda = 0, \quad \mu \equiv \beta\delta_{\{t=0\}}, \quad p \equiv -\beta\zeta \quad \text{with } \zeta \in \partial_x^{>} h(0,x_0) \text{ and some } \beta > 0, \tag{14}$$

satisfies (13) but violates (12) and is therefore excluded.

Variants of this theorem are easily proved. For example, a version of the theorem in which the functional inequality state constraint $h(t,x(t)) \leq 0$ is replaced by a set inclusion $x(t) \in X(t)$ for some upper semicontinuous multifunction $X : [0,1] \rightrightarrows \mathbb{R}^n$ can be derived by expressing this latter constraint as a functional inequality constraint with

$$h(t,x) = d_{X(t)}(x)$$

and applying Thm. 2.1 (c.f. [3, Chapter 3]). (Here $d_{X(t)}(x)$ denotes the distance function of the point x to the set $X(t)$, $d_{X(t)}(x) := \inf\{\|x-y\| : y \in X(t)\}$.)

Under the stated hypotheses, the theorem excludes one kind of multiplier triviality, namely (14). However it still allows $\lambda = 0$. It can be shown that, if we assume $\bar{x}(1) \in \text{int}\, C$ and also strengthen (CQ) to require the existence of $\delta > 0$ and $\epsilon > 0$ such that for a.e. $t \in [0,1]$

$$\zeta \cdot [f(t,\bar{x}(t),\tilde{u}(t)) - f(t,\bar{x}(t),\bar{u}(t))] < -\delta$$

for all $\zeta \in \partial_x^{>} h(s,\bar{x}(s))$ and $s \in \{\tau \in [0,1] : h(\tau,\bar{x}(\tau)) = 0\} \cap [t-\epsilon, t+\epsilon]$, then the multipliers (λ, p, μ) whose existence is asserted in Thm. 2.1, must satisfy $\lambda > 0$. Thus, implicit in Thm. 2.1 is a *normal* form of the Maximum Principle, valid under stronger hypotheses.

3 Proof of the Results

In what follows we shall assume that $h(0,x_0) = 0$, since, otherwise, the conditions of Thm. 2.1 cannot be satisfied by the trivial multipliers (5).

Choose $\alpha \in (0,1]$. Consider now measurable functions v, and absolutely continuous functions x satisfying

$$(S)\begin{cases} \dot{x}(t) = f(t,x(t),\bar{u}(t)) + v(t)\cdot\Delta f(t,x(t)) & \text{a.e. } t \in [0,\alpha] \\ x(0) = x_0 & \\ x(t) \in \bar{x}(t) + \delta'\mathbb{B} & \text{all } t \in [0,\alpha] \\ v(t) \in \{0\}\cup\{1\} & \text{a.e. } t \in [0,\alpha] \end{cases} \tag{15}$$

where we define

$$\Delta f(t,x) := f(t,x,\tilde{u}(t)) - f(t,x,\bar{u}(t)). \tag{16}$$

Here $\tilde{u}$ is the control function featuring in the constraint qualification (CQ)

The key idea of the proof is to replace the original control problem by one in which the state constraint is eliminated on $[0,\alpha]$, for arbitrary small α. The multipliers for this new problem are nondegenerate. We then obtain a set of multipliers for the original problem by passing to the limit $\alpha \downarrow 0$. Our construction is of such a nature that the limiting multipliers are nondegenerate.

The following Lemma, 3.1, stated without proof, is a simple consequence of the hypotheses imposed on the data and standard Gronwall-type estimates.

Lemma 3.1 *Consider a pair of functions (x,v) solving the system of equations (S), and $\bar{x}$ solving (P). There exist positive constants A, and B such that for α small enough*

$$\begin{aligned} \|x(t) - x_0\| &\leq At, \\ \|x(t) - \bar{x}(t)\| &\leq B\int_0^t v(s)ds \end{aligned}$$

for all $t \in [0,\alpha]$.

The following lemma is of key importance. It establishes that every trajectory x associated with the system of equations (S) satisfies the state constraint on some initial interval of time.

Lemma 3.2 *By reducing the size of α if necessary we can ensure that*

$$h(t,x(t)) \leq 0 \quad \textit{for all } t \in [0,\alpha], \tag{17}$$

for all trajectories x solving system (S).

Proof. Choose an α satisfying

$$\alpha < \min\left\{\frac{\delta}{8K_hK_f(A+B)}, \frac{\epsilon_1}{A}, \epsilon\right\}. \tag{18}$$

Suppose, in contradiction, that for some fixed $t \in [0, \alpha]$

$$h(t, x(t)) > 0. \tag{19}$$

Define for $\beta \in [0, 1]$

$$r(\beta) := h(t, \bar{x}(t) + \beta(x(t) - \bar{x}(t))).$$

In view of the properties of h as a function of x, r is continuous. We have also that

$$\begin{aligned} r(0) &= h(t, \bar{x}(t)) \leq 0, \\ r(1) &= h(t, x(t)) > 0. \end{aligned}$$

It follows that the set

$$D := \{\beta \in [0, 1] : r(\beta) = 0\}$$

is non-empty, closed and bounded. We can therefore define

$$\beta_m := \max_{\beta \in D} \beta.$$

Since $r(1) > 0$, we have $\beta_m < 1$. Take any $\beta \in (\beta_m, 1]$.

Applying the Lebourg Mean-Value Thm. ([3]), we obtain

$$\begin{aligned} h(t, x(t)) - r(\beta) &= \zeta_t \cdot [x(t) - \bar{x}(t) - \beta(x(t) - \bar{x}(t))] \\ &= (1 - \beta)\zeta_t \cdot [x(t) - \bar{x}(t)] \end{aligned}$$

for some $\zeta_t \in \operatorname{co} \partial_x h(t, \hat{x})$, and $\hat{x}$ in the segment $(x(t), \bar{x}(t) + \beta[x(t) - \bar{x}(t)])$. As $r(\beta) > 0$ for all $\beta \in (\beta_m, 1]$, we have that $h(t, \hat{x}) > 0$, which implies that $\operatorname{co}\partial_x h(t, \hat{x}) \subset \partial_x^{>} h(t, \hat{x})$. It follows that $\zeta_t \in \partial_x^{>} h(t, \hat{x})$.

Expanding the expression above yields

$$
\begin{aligned}
&h(t,x(t)) - r(\beta)\\
&\quad = (1-\beta)\,\zeta_t \cdot \int_0^t [f(s,x(s),\bar{u}(s)) + v(s)\Delta f(s,x(s))\\
&\qquad\qquad -f(s,\bar{x}(s),\bar{u}(s))]\,ds\\
&\quad \le (1-\beta)\left(\zeta_t \cdot \int_0^t v(s)\Delta f(s,x(s))\,ds\ + \|\zeta_t\| K_f \int_0^t \|x(s)-\bar{x}(s)\|\,ds\right)\\
&\quad \le (1-\beta)\left(\int_0^t v(s)\zeta_t \cdot \Delta f(s,x_0)\,ds\ + 2K_f\|\zeta_t\| \int_0^t v(s)\|x(s)-x_0\|\,ds\right.\\
&\qquad\qquad \left. + K_h K_f \int_0^t \|x(s)-\bar{x}(s)\|\,ds\right)\\
&\quad \le (1-\beta)\left(-\delta \int_0^t v(s)\,ds\ + 2K_f K_h A t \int_0^t v(s)\,ds\right.\\
&\qquad\qquad \left. + K_h K_f B \int_0^t \int_0^s v(\tau) d\tau\,ds\right)\\
&\quad \le (1-\beta)\,(-\delta + K_h K_f (2A+B)t) \int_0^t v(s) ds\\
&\quad \le 0 \quad \text{for all } \beta \in (\beta_m, 1].
\end{aligned}
$$

Here we have used the fact that the norm of every element of the subdifferential is bounded by the Lipschitz rank of the function. In the last two inequalities we have used (CQ) and (18).

Since r is continuous and $r(\beta_m) = 0$ it follows that

$$h(t,x(t)) \le 0.$$

This contradicts (19). The proof is complete. □

Take a decreasing sequence $\{\alpha_i\}$ on $(0,\alpha)$, converging to zero. Associate with each α_i the following problem (P_i), in which satisfaction of the state constraint is enforced only on the subinterval $[\alpha_i, 1]$.

$$(P_i)\quad \text{Minimize} \quad g(x(1)) \tag{20}$$

subject to

$$\dot{x}(t) = f(t,x(t),\bar{u}(t)) + v(t)\cdot \Delta f(t,x(t)) \quad \text{a.e. } t \in [0,\alpha_i)$$

$$\dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. } t \in [\alpha_i, 1] \tag{21}$$
$$x(0) = x_0$$
$$x(1) \in C$$
$$u(t) \in \Omega(t) \quad \text{a.e. } t \in [\alpha_i, 1]$$
$$v(t) \in \{0\} \cup \{1\} \quad \text{a.e. } t \in [0, \alpha_i)$$
$$h\left(t, x(t)\right) \leq 0 \quad \text{for all } t \in [\alpha_i, 1]. \tag{22}$$

Note that we can write the first dynamic equation as

$$\dot{x}(t) = f\left(t, x(t), \hat{u}(t)\right) \quad \text{a.e. } t \in [0, \alpha_i)$$

where

$$\hat{u}(t) = \begin{cases} \bar{u}(t) & \text{if } v(t) = 0 \\ \tilde{u}(t) & \text{if } v(t) = 1. \end{cases}$$

The function $\hat{u}$ is a measurable function and $\hat{u}(t) \in \Omega(t)$. These facts combine with the previous lemma to ensure that all admissible state trajectories x for (P_i) such that $\|x(t) - \bar{x}(t)\|_{L^\infty} < \delta'$ are contained in the set of admissible trajectories of (P). Moreover the process $(x, (u, v)) \equiv (\bar{x}, (\bar{u}, 0))$ for (P_i) has cost identical to that of (P). We have proved the following Lemma.

Lemma 3.3 *For each i, the process $(\bar{x}, (\bar{u}, 0))$ is a strong local minimizer for (P_i).*

Now we apply a strengthened version of the Maximum Principle in [3] to the strong local minimizer $(\bar{x}, (\bar{u}, 0))$ for (P_i), in which $\partial^{>} h$ replaces a coarser hybrid subgradient used in [3] and the transversality conditions are expressed in terms of the limiting normal cone and limiting subdifferential in place of their convex hulls. The modifications to the analysis to achieve those refinements are indicated in [7].

These necessary conditions for problem (P_i) assert the existence of an arc $p_i : [0, 1] \mapsto \mathbb{R}^n$, a measurable function γ_i, a nonnegative Radon measure $\mu_i \in C^*([\alpha_i, 1], \mathbb{R})$, and a scalar $\lambda_i \geq 0$ such that

$$\mu_i\{[\alpha_i, 1]\} + ||p_i|| + \lambda_i > 0, \tag{23}$$

$$-\dot{p}_i(t) \in \begin{cases} \operatorname{co}\partial_x \left(p_i(t)\cdot f(t,\bar{x}(t),\bar{u}(t))\right) & \text{a.e. } t\in[0,\alpha_i) \\ \operatorname{co}\partial_x \left(\left(p_i(t)+\int_{[\alpha_i,t)}\gamma_i(s)\mu_i(ds)\right)\right. & \\ \qquad \left.\cdot f(t,\bar{x}(t),\bar{u}(t))\right) & \text{a.e. } t\in[\alpha_i,1] \end{cases} \tag{24}$$

$$-\left(p_i(1)+\int_{[\alpha_i,1]}\gamma_i(s)\mu_i(ds)+\lambda_i\xi_i\right)\in N_C(\bar{x}(1)) \tag{25}$$

where $\xi_i \in \partial_x g(\bar{x}(1))$,

$$\gamma_i(t)\in \partial_x^{>} h(t,\bar{x}(t)) \quad \mu_i\text{-a.e. }, \tag{26}$$

$$\operatorname{supp}\{\mu_i\}\subset\{t\in[\alpha_i,1]: h\left(t,\bar{x}(t)\right)=0\}\,, \tag{27}$$

for almost every $t\in[0,\alpha_i)$, $v(t)=0$ maximizes over $\{0\}\cup\{1\}$

$$\nu\mapsto \nu\, p_i(t)\cdot[f(t,\bar{x}(t),\tilde{u}(t))-f(t,\bar{x}(t),\bar{u}(t))], \tag{28}$$

and for almost every $t\in[\alpha_i,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u\mapsto \left(p_i(t)+\int_{[\alpha_i,t)}\gamma_i(s)\mu_i(ds)\right)\cdot f\left(t,\bar{x}(t),u\right). \tag{29}$$

It remains to pass to the limit as $i\to\infty$ and thereby to obtain a set of nondegenerate multipliers for the original problem.

Without changing the notation, we extend μ_i as a regular Borel measure on $[0,1]$

$$\mu_i(B)=\mu_i(B\cap[\alpha_i,1]) \quad \text{for all Borel set } B\subset[0,1].$$

Extend also γ_i, originally defined on $[\alpha_i,1]$, arbitrarly to the interval $[0,1]$ as a Borel measurable function.

With these extensions, noting that $\mu([0,\alpha_i))=0$, we can write

$$-\dot{p}_i(t)\in\operatorname{co}\partial_x\left(\left(p_i(t)+\int_{[0,t)}\gamma_i(s)\mu_i(ds)\right)\cdot f(t,\bar{x}(t),\bar{u}(t))\right) \quad \text{a.e. } t\in[0,1]$$

By scaling the multipliers we can then ensure that

$$\|p_i\| + \|\mu_i\| + \lambda_i = 1.$$

The multifunction $\partial_x^{>} h$ is uniformly bounded, compact, convex, and has a closed graph. As $\{p_i\}$ is uniformly bounded and $\{\dot{p}_i\}$ is uniformly integrally bounded, we can arrange by means of subsequence extraction [3, Thm. 3.1.7, Prop. 3.1.8] that

$$p_i \to p \text{ uniformly}, \quad \gamma_i d\mu_i \to \gamma d\mu \text{ weak}^*, \quad \lambda_i \to \lambda, \quad \xi_i \to \xi$$

where μ is the weak* limit of μ_i, γ is a measurable selection of $\partial_x^{>} h(t, \bar{x}(t))$ μ a.e., and $\xi \in \partial g(\bar{x}(1))$. To obtain ξ we have used the fact that $\partial g(\bar{x}(1))$ is a compact set.

It follows that the conditions (10), (12), and (7) for problem (P) are satisfied and as $N_C(\bar{x}(1))$ is closed (8) also holds.

Consider the set $S_i = [\alpha_i, 1] \backslash \Omega_i$ where Ω_i is a null Lebesgue measure set in $[\alpha_i, 1]$ containing all times where the maximization of (29) is not achieved at $\bar{u}$. We can then write

$$\left(p_i(t) + \int_{[\alpha_i, t)} \gamma_i(s) \mu_i(ds) \right) \cdot f\left(t, \bar{x}(t), u\right) \leq \\ \left(p_i(t) + \int_{[\alpha_i, t)} \gamma_i(s) \mu_i(ds) \right) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right),$$

for all $t \in S_i$ and for all $u \in \Omega(t)$.

Now consider the full measure set $S = (0, 1] \backslash \bigcup_i \Omega_i$. Fix some t in S. Then for all $i > N$, where N is such that $\alpha_N \leq t$ we have

$$\left(p_i(t) + \int_{[0, t)} \gamma_i(s) \mu_i(ds) \right) \cdot f\left(t, \bar{x}(t), u\right) \leq \\ \left(p_i(t) + \int_{[0, t)} \gamma_i(s) \mu_i(ds) \right) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right).$$

for all $u \in \Omega(t)$. Applying limits to both sides of this inequality we obtain (11).

At this point we have established that the set of multipliers (p, μ, λ), obtained as a limit of the subsequence (p_i, μ_i, λ_i) satisfies the necessary conditions of optimality for the original problem (P).

Finally, we verify

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda > 0. \tag{30}$$

In view of the constraint qualification, there exists a constant $\delta > 0$ such that, for all i and for a.e. $t \in [0, \alpha_i)$,

$$\zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta,$$

for all $\zeta \in \partial_x^> h(s, x),\ s \in [0, \epsilon),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

Suppose, in contradiction, that

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda = 0. \tag{31}$$

Since $(\lambda, \mu, p) \neq 0$, we must have

$$\begin{aligned} &\lambda = 0, \\ &\mu = \beta\delta_{\{0\}}, \\ &p(t) = -\beta\zeta \quad \text{for some } \beta > 0 \text{ and } \zeta \in \partial_x^> h(0, x_0). \end{aligned} \tag{32}$$

The constraint qualification (CQ) implies

$$-p(t) \cdot \Delta f(t, x_0) = \beta\zeta \cdot \Delta f(t, x_0) < -\delta\beta \quad \text{a.e. } t \in [0, \alpha_i).$$

On the other hand the maximization condition on v (28) implies that

$$p_i(t) \cdot [f(t, \bar{x}(t), \tilde{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] \leq 0 \quad \text{a.e. } t \in [0, \alpha_i). \tag{33}$$

But expanding this last expression we can write

$$\begin{aligned} &p_i(t) \cdot \Delta f(t, \bar{x}(t)) \\ &\quad = p(t) \cdot \Delta f(t, x_0) + (p_i(t) - p(t))\Delta f(t, x_0) + p_i(t)[\Delta f(t, \bar{x}(t)) - \Delta f(t, x_0)] \\ &\quad \geq \delta\beta - 2K_u\|p_i(t) - p(t)\| - 2K_f\|\bar{x}(t) - x_0\|\|p_i(t)\| \\ &\quad \geq \delta\beta - 2K_u\|p_i(t) - p(t)\| - 2K_f A t\|p_i(t)\| \end{aligned}$$

By the uniform convergence of p_i, we can make $\|p_i - p\| < \bar{\epsilon}$ for any $\bar{\epsilon} > 0$ of our choice provided we choose a sufficiently large i. Moreover $\|p_i\| \leq 1$.

It follows that

$$p_i(t) \cdot \Delta f(t, \bar{x}(t)) \geq \delta\beta - 2K_u\bar{\epsilon} - 2K_f A\alpha_i > \delta\beta/2 > 0$$

if $\bar{\epsilon} < \delta\beta/(8K_u)$ and $\alpha_i < \delta\beta/(8K_f A)$.

So, we would have $p_i(t) \cdot \Delta f(t, \bar{x}(t)) > 0$ for a.e. $t \in [0, \alpha_i)$ contradicting equation (33). We deduce (30).

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