Dynamic Pricing and Inventory Control with Learning

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Abstract: Optimal operating policies and corresponding managerial insight are developed for the decision problem of coordinating supply and demand when (i) both supply and demand can be influenced by the decision maker and (ii) learning is pursued. In particular, we determine optimal stocking and pricing policies over time when a given market parameter of the demand process, though fixed, initially is unknown. Because of the initially unknown market parameter, the decision maker begins the problem horizon with a subjective probability distribution associated with demand. Learning occurs as the firm monitors the market’s response to its decisions and then updates its characterization of the demand function. Of primary interest is the effect of censored data since a firm’s observations often are restricted to sales. We find that the first-period optimal selling price increases with the length of the problem horizon. However, for a given problem horizon, prices can rise or fall over time, depending on how the scale parameter influences demand. Further results include the characterization of the optimal stocking quantity decision and a computationally viable algorithm. © 2002 Wiley Periodicals, Inc. Naval Research Logistics 49: 303–325, 2002; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/nav.10013

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1. INTRODUCTION

In developing inventory models it typically is assumed that the parameters that define the demand process are exogenous. As a result, the primary objective of these models is to determine a strategy for matching supply to the given demand process so as to achieve operational efficiency. There basically are two cases. If the given process is known with certainty, then either the economic order quantity model or the Wagner–Whitin model constitutes the fundamental building block for analysis, depending on whether the application is continuous time or discrete time. If the given process is not known with certainty, then the newsvendor model provides the fundamental building block. Such uncertainty could exist because the process is inherently stochastic, the process is deterministic but one or more of the defining parameters is unknown to the decision maker, or the process is a combination of both. Regardless, if uncertainty does exist, then the decision maker should consider the possibility of learning, that is, refining over time the characterization of the uncertainty as new information is ascertained.

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Our aim in this paper is to develop insight into the decision problem of coordinating supply and demand when (i) both supply and demand can be influenced by the decision maker and (ii) learning is pursued. In particular, we examine the effect on the stocking and pricing decisions over time when a given market parameter of the demand process, though fixed, initially is unknown to the decision maker. We assume that demand in each of $T$ periods is a deterministic function of price, but because one of the market parameters initially is unknown, the decision maker begins the problem horizon with a subjective probability distribution function associated with demand. As an immediate consequence, the decision maker behaves initially as if demand were drawn from a suitably specified stochastic process.

However, by observing sales on an ongoing basis, the decision maker updates the characterization of uncertainty until either the unknown parameter is revealed or the problem horizon ends. If, in any period, the unknown parameter is revealed, then the decision maker operates under certainty in all subsequent periods. But such an event occurs if and only if leftovers remain at the end of a period. This is because if leftovers remain, then demand is observed, which implies that the unknown parameter is deduced, and if the unknown parameter is revealed, then demand becomes known as a function of price, which implies that leftovers will not occur again. In the meantime, for each period that leftovers do not occur, the decision maker’s observation of sales results in a censored observation of demand. Nevertheless, this information is used to update the subjective probability distribution characterizing the unknown parameter of the demand function. The resulting multiperiod model formulation is a variant of the newsvendor model with price effects (Whitin [36]); its analysis yields insight into the dynamics of the stocking and pricing decisions made by the firm.

In our model, by assuming a deterministic, albeit initially unknown, relationship between demand and price, we explicitly allow for the possibility that all uncertainty can be eliminated through learning. Conceivably, an additional layer of uncertainty, inherent noise, also could exist. Inherent noise injects an additional degree of difficulty into our model because it obscures learning with respect to the assignable form of uncertainty described above. For this reason, we restrict our attention to a learning model that does not include an unassignable component of uncertainty. This allows us to develop structural results and obtain insight regarding the optimal stocking and pricing behavior of the firm under robust specifications of the price–demand relationship.

Related learning models in operations management assume price is exogenous. Such models typically assume that the demand distribution is refined over time after the observation of either demand (e.g., Scarf [31, 32], Karlin [17], Azoury [3], Lovejoy [21], Murray and Silver [24], Hausman and Peterson [15], Crowston, Hausman, and Kampe [8], Bitran, Haas, and Matsuo [5], Matsuo [22]) or sales (e.g., Krouse and Senchack [18], Harpaz, Lee, and Winkler [14], Braden and Freimer [6], Reyniers [30], Lariviere and Porteus [19], Nahmias [25], Agrawal and Smith [1]). Due to the complexity arising from the incorporation of censored data, it is customary to assume a perishable product when updates are based on the observation of sales. The economics and marketing literatures also include related models (e.g., Grossman, Kihlstrom, and Mirman [13], Balvers and Cosimano [4], Lazear [20], Trefler [35], Braden and Oren [7]), but these models assume that the stocking level is established after demand is observed, in effect making that decision an exogenous parameter.

Existing dynamic models including both stocking and pricing as decisions do not incorporate learning (Federgruen and Heching [11], Ernst [10], Zabel [37], Thowsen [34]). An exception is Alpern and Snower [2], who analyze a problem similar to the one developed in this paper, but use a different construct. Whereas we use a subjective probability distribution to characterize the initial uncertainty in the price-dependent demand function, they assume only upper and lower bounds on the function. Learning in their model is based on a high-low search methodology as
the firm tries to hone in on the true form of demand. Their algorithm maximizes the present value of the profit stream over time, balancing the benefit of having a lower production level to save on holding costs with the benefit of having a higher production level to increase the chance of gaining more information about demand.

We follow the lead of Krouse and Senchack [18] in formulating our problem by assuming that the product is perishable and that demand is a deterministic function, including an initially unknown parameter characterized by a subjective distribution that is updated over time. One implication of this formulation is that the only link between subsequent periods is information, which allows us to focus on how the potential benefit of learning affects decision making without having the results blurred by the complementary potential benefit associated with carrying inventory into subsequent periods. Another implication is that the dynamic nature of the problem ends the first time leftovers occur, with all subsequent decisions based on full information. This provides a mechanism for measuring the potential benefit of learning and for establishing structural insights into such benefits, particularly when different forms for the demand function are considered. Our analysis extends the one by Krouse and Senchack [18] in two ways: We include pricing as a decision, and we do not limit the analysis to the special case in which the subjective distribution for demand uncertainty is characterized by the exponential probability density function (p.d.f.).

In Section 2, we develop the general dynamic program for the problem and define two alternative formulations of the demand function: the additive demand case and the multiplicative demand case. We identify and analyze structural properties of the optimal solution in Section 3. Then, we demonstrate the solution procedure for reducing the computational complexity of the problem to a single-variable search in Section 4. In Section 5, we illustrate the solution procedure and demonstrate the structure features of the results with numerical examples. In Section 6, we formulate and briefly discuss the extension in which the assumption of a perishable product is relaxed. In Section 7 we provide closing remarks. Proofs of all lemmas and theorems are in the Appendix.

2. DYNAMIC PROGRAMMING FORMULATION

Consider a price-setting firm that stocks a perishable product, faces a price-dependent demand function that is deterministic but initially includes an unknown parameter, and has the objective of determining jointly a stocking quantity \( q \) and selling price \( p \) for each of \( T \) periods to maximize the total expected discounted profit. The per-unit purchase cost of a unit is \( c \), and the per-unit disposal cost/salvage value of a leftover is \( h \) (\( h \geq -c \), so that per-unit salvage value is not greater than per-unit purchase cost). The unknown parameter in demand is independent of \( p \) and is modeled as either an additive term (Mills [23]) or a multiplicative term (Karrlin and Carr [17]):

\[
D(p, \epsilon) = y(p) + \epsilon \quad \text{or} \quad D(p, \epsilon) = y(p)\epsilon,
\]

respectively, where \( y(p) \) is a decreasing function that captures the deterministic relationship between demand and price and \( \epsilon \) is a random variable. Three forms for \( y(p) \) commonly appear in the literature:

\[
\begin{align*}
y(p) &= a - bp \quad (a, b > 0), \quad (1a) \\
y(p) &= ae^{-bp} \quad (a, b > 0), \quad (1b) \\
y(p) &= ap^{-b} \quad (a > 0; b > 1). \quad (1c)
\end{align*}
\]

We use all three specifications when considering the multiplicative demand case. However, it is somewhat unnatural to use either (1b) or (1c) in the additive demand case because, then,
$D(p, \epsilon) > 0$ even as $p \to \infty$ if it turns out that $\epsilon > 0$. Under such circumstances, infinite profit is possible. Consequently, when considering the additive demand case, we use (1a) only. Given these forms for $y(p)$, the unknown parameter $\epsilon$ can be interpreted as uncertainty in the precision with which the market potential is known.

One useful property that follows from (1) is that the function $(p - c)y(p)$ is quasiconcave. We define the value of $p$ that maximizes this function as $p^0(c)$ and we record in Table 1 $p^0(c)$ for each specification of $y(p)$. Moreover, we assume that $p^0(-h) + h \geq 0$, where we define $p^0(-h)$ analogously to $p^0(c)$. This condition simply means that we require the function $(p + h)y(p)$ to be nonnegative when evaluated at its maximum; it is satisfied if $y(p)$ is specified by (1a) or (1b), or if $y(p)$ is specified by (1c) and $h \leq 0$.

Let $F(\cdot)$ represent the initial subjective cumulative distribution function (c.d.f.) assigned to $\epsilon$ and define $f(\cdot) = dF(\cdot)$ as the corresponding initial p.d.f. We assume that $F(\cdot)$ has a nondecreasing failure rate, where $r(\cdot) = f(\cdot)/[1 - F(\cdot)]$ denotes the failure rate. In order to assure that $D(p, \epsilon) > 0$ for some range of $p$, we require an initial lower bound on $\epsilon: \epsilon \geq A$, where $A > -a$ in the additive demand case and $A > 0$ in the multiplicative demand case. In addition, $\epsilon$ may or may not have a finite upper bound $B$.

At the beginning of a new period, either uncertainty no longer exists and the problem is deterministic or the subjective distribution is updated, depending on the information state and the outcome of events associated with the previous period. If no uncertainty existed in the previous period, then the value of $\epsilon$ already has been revealed and the decision scenario remains a deterministic one at the beginning of the new period. However, if uncertainty did exist in the previous period, then the information state at the beginning of the new period depends on whether or not leftovers occurred in the previous period. If there were leftovers, then demand is observed, which implies that the actual value of $\epsilon$ is deduced and the decision scenario at the beginning of the new period becomes a deterministic one. If there were no leftovers, then only a lower bound for demand is observed, which implies that a new lower bound for $\epsilon$ is deduced and the decision scenario at the beginning of the new period remains nondeterministic, albeit the subjective distribution characterizing the uncertainty can be revised. In particular, whenever a new lower bound for $\epsilon$ is determined, the subjective p.d.f. can be updated by truncating the initial p.d.f. at the new lower bound and renormalizing. Thus, letting $x$ represent a newly determined lower bound for $\epsilon$, we can write the updated subjective p.d.f. for $\epsilon$ as a function $x$, which we define as $g(\cdot, x)$, as follows:

$$g(\cdot, x) = \frac{f(\cdot)}{1 - F(x)}.$$  

(2)

Correspondingly, we define $G(\cdot, x) = [F(\cdot) - F(x)]/[1 - F(x)]$ as the updated c.d.f.

To summarize, the problem is dynamic because the information state corresponding to the unknown parameter $\epsilon$ changes over time. Given that $\epsilon$ is unknown at the beginning of a given

<table>
<thead>
<tr>
<th>$y(p)$</th>
<th>Parameter restrictions</th>
<th>$p^0(\omega) \equiv \arg\max{(p - \omega)y(p)}$</th>
<th>Related assumption</th>
<th>For use in $^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a - bp$</td>
<td>$a &gt; 0$, $b &gt; 0$</td>
<td>$(a + bw)/2b$</td>
<td>$-$</td>
<td>A, M</td>
</tr>
<tr>
<td>$ae^{-bp}$</td>
<td>$a &gt; 0$, $b &gt; 0$</td>
<td>$(1 + bw)/b$</td>
<td>$-$</td>
<td>M</td>
</tr>
<tr>
<td>$ap^{-b}$</td>
<td>$a &gt; 0$, $b &gt; 1$</td>
<td>$b/a(b - 1)$</td>
<td>$h \leq 0$</td>
<td>M</td>
</tr>
</tbody>
</table>

$^a$ “A” refers to the additive demand case; “M” refers to the multiplicative demand case.
period and \( p \) and \( q \) represent the firm’s decisions for the period, if it is observed that \( D(p, \epsilon) > q \), then the information state changes because a new lower bound for \( \epsilon \) is determined [the new lower bound is \( q - y(p) \) for the additive demand case or \( q/y(p) \) for the multiplicative demand case], and the subjective distribution is revised accordingly. However, if under similar circumstances, it is observed that \( D(p, \epsilon) \leq q \), then \( \epsilon \) is revealed and full information prevails, which implies that the dynamics end.

Since the problem dynamics are established through the updating procedure and the updating procedure relies solely on determining a revised lower bound for \( \epsilon \), which itself depends on a prescribed relationship between \( p \) and \( q \), it is useful to think in terms of the prescribed relationship rather than in terms of \( p \) and \( q \). To that end, we define the following surrogate decision variable:

\[
z \equiv \begin{cases} 
q - y(p) & \text{for the additive demand case,} \\
q/y(p) & \text{for the multiplicative demand case.}
\end{cases}
\]  

The convenience of \( z \) is in the separability that it establishes. Without \( z \) there are two links between subsequent periods, one each through the two decision variables \( p \) and \( q \). But with \( z \), both links are severed and replaced with a single link established through the surrogate decision \( z \) [see (4) below]. This is because (i) if an update of the subjective distribution is required, \( z \) serves as a sufficient statistic for making the update and (ii) an update is required only if no leftovers occur at the end of a period, which is an event that depends only on \( z \) [from (3), leftovers occur if and only if \( z > \epsilon \)].

An alternative tactic of reducing the double link to a single link by severing only one of the original links while maintaining the other is a more formidable task. Since, originally, links exist both through \( p \) and through \( q \), solving, for example, the sequence of \( q \)-decisions as a function of the sequence of \( p \)-decisions and thereby reducing the problem to a single decision space would require solving a dynamic program. In contrast, however, with the \( z \) construct, any effect on the future resulting from the current decision for \( p \) or \( q \) is captured through \( z \). Thus, the entire decision problem can be reduced to a single decision space (\( z \)-space) simply by substituting for either \( p \) or \( q \) using (3) and then solving a single period problem in order to cast the other (\( p \) or \( q \)) as a function of \( z \) [see (7) below].

We now develop a generic dynamic program to represent the joint stocking and pricing optimization problem using the transformation of decision variables given by (3). Let \( z_t \) denote the conditionally optimal value for \( z \) in period \( t \), given that demand is not observed before the beginning of period \( t \). If demand is observed before the beginning of period \( t \), then the dynamic nature of the problem ends and \( z_t \) has no meaning. In addition, let

\[
V_t(u) = \text{maximum discounted profit from period } t \text{ to } T \text{ under full information, given that it is known that } \epsilon = u,
\]

\[
W_t(x) = \text{maximum expected discounted profit from period } t \text{ to } T \text{ under uncertainty, given that it is known at the beginning of period } t \text{ that } \epsilon \geq x,
\]

\[
\Pi(z, p, x) = \text{the single-period expected contribution, given that it is known at the beginning of the period that } \epsilon \geq x; \text{ and } p \text{ and } z \text{ are the period’s decision variables. [Note: our choice to use (3) here to substitute for } q \text{ is not arbitrary; although substituting for } p \text{ also is viable, it requires taking the inverse of } y(p).]}
\]
Then,

\[ W_t(x) = \max_{z,p} \left\{ \Pi(z,p,x) + \alpha \left[ \int_z^z V_{t+1}(u)g(u,x) \, du + \int_z^\infty W_{t+1}(z)g(u,x) \, du \right] \right\}, \quad (4) \]

where \( V_{T+1}(u) = W_{T+1}(z) \equiv 0 \) and \( \alpha \) is the discounting factor. The expression for the single-period expected contribution depends on whether demand uncertainty is additive or multiplicative, but it can be written in a general form as follows (Petruzzi and Dada [27]):

\[
\Pi(z,p,x) = (p-c)(E[D(p,\epsilon)]|x] - E[Shortages(z,p,\epsilon)|x]) - (c+h)E[Leftovers(z,p,\epsilon)|x]
\]

By applying the definitions listed in Table 2, we obtain, after some algebra,

\[
\Pi(z,p,x) = (p-c)D(p,x) + M(p)\{[p-c](\mu(x) - x - \Theta(z,x)) - (c+h)\Lambda(z,x)\}. \quad (5)
\]

With full information, if it is known that \( \epsilon = u \), then there is no risk of having leftovers or shortages. The optimal course of action in such a case is to choose the price that maximizes \( (p-c)D(p,u) \) and set the stocking quantity equal to the corresponding deterministic demand. Thus

\[
V_{t+1}(u) = \frac{1 - \alpha^{T-t}}{1 - \alpha} [p(u) - c]D(p(u),u), \quad \text{where } p(u) = \text{argmax}_p \{(p-c)D(p,u)\}. \quad (6)
\]

Given (1) and the related boundary conditions from Table 1, it is easy to verify that \( p(u) \) is determined uniquely as a function of \( u \), both for the additive demand case and the multiplicative demand case.

Referring to (4), neither \( V_{t+1}(u) \) nor \( W_{t+1}(z) \) depends on \( p \). Consequently, the optimal price for each period can be written as a function of \( z \) and \( x \) by solving a single-period problem for fixed values of \( z \) and \( x \). This reduces the dynamic program to one that requires only a single maximization at each stage:

\[
W_t(x) = \max_{z,x} \left\{ \Pi(z,x) + \alpha \left[ \int_x^z V_{t+1}(u)g(u,x) \, du + \int_z^\infty W_{t+1}(z)g(u,x) \, du \right] \right\}, \quad (7)
\]

where \( \Pi(z,x) \equiv \Pi(z,p_{zx},x) \) and \( p_{zx} = \text{argmax}_p \Pi(z,p,x) \), given \( z \) and \( x \). Given (1), it can be shown that the maximand over \( p \) of \( \Pi(z,p,x) \) is not at a boundary, for given values of \( z \) and \( x \). Consequently, each pair of values for \( z \) and \( x \) provides a corresponding selling price, \( p_{zx} \), and a corresponding stocking quantity, \( q_{zx} \), where \( p_{zx} \) satisfies \( \partial \Pi(z,p,x)/\partial p = 0 \) and

**Table 2. Definitions.**

<table>
<thead>
<tr>
<th>( \mu(x) )</th>
<th>( \equiv \int_x^\infty u g(u,x) , du )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda(z,x) )</td>
<td>( \equiv \int_x^z (z-u) g(u,x) , du )</td>
</tr>
<tr>
<td>( \Theta(z,x) )</td>
<td>( \equiv \int_x^\infty (u-z) g(u,x) , du )</td>
</tr>
<tr>
<td>( M(p) )</td>
<td>( \begin{cases} 1 &amp; \text{for the additive demand case} \ y(p) &amp; \text{for the multiplicative demand case} \end{cases} )</td>
</tr>
</tbody>
</table>

Table 3. Summary of results.

<table>
<thead>
<tr>
<th>ID</th>
<th>Variable</th>
<th>Additive demand case</th>
<th>Multiplicative demand case</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>$p_t$</td>
<td>Strictly increasing (A)</td>
<td>Strictly decreasing (E)</td>
</tr>
<tr>
<td>R2</td>
<td>$q_t$</td>
<td>Either strictly increasing or increasing-decreasing (E)</td>
<td>Strictly increasing (A)</td>
</tr>
<tr>
<td>R3</td>
<td>$p_{1,T}$</td>
<td>Strictly increasing (A)</td>
<td>Strictly increasing (A)</td>
</tr>
<tr>
<td>R4</td>
<td>$q_{1,T}$</td>
<td>Strictly increasing (A)</td>
<td>Either strictly increasing or increasing-decreasing (A)</td>
</tr>
</tbody>
</table>

$^a$This is an analytical result given that R1 is true for the multiplicative demand case.

$q_{xx}$ is constructed from $z$ and $p_{xx}$ using (3). If we define $z_t$ as the solution to the maximization associated with stage $t$ of the dynamic program, which consequently becomes the state variable for stage $t + 1$, then the conditionally optimal selling price for period $t$, given that demand is not observed before the beginning of period $t$, is $p_t = p_{z_t,z_{t-1}}$. The corresponding conditionally optimal stocking quantity is $q_t = q_{z_t,z_{t-1}}$. If in any period demand is observed, the dynamic nature of the problem ends and the stocking and pricing policy becomes stationary, set in accordance with having full information.

3. STRUCTURE OF THE OPTIMAL POLICY

In this section, we develop analytical properties of the optimal stocking and pricing policy, categorizing the results along two dimensions. First we investigate how the optimal policy would evolve over time if demand were not observed by the prespecified time $T$. Then we consider how the optimal policy for the first period changes as $T$ changes. We find that a certain symmetry exists between the additive and the multiplicative demand cases, but, in both cases, the results are predicated on Theorems 1 and 2. Theorem 1 establishes that $z_t$, the optimal value for $z$ in period $t$ given that demand has not been observed prior to period $t$, is strictly increasing in $t$. Theorem 2 establishes that $z_{1,T}$, the optimal value for $z$ in period 1 of a $T$-period problem given that $A$ is the initial lower bound for $z$, is strictly increasing in $T$. Implications of these two theorems differ between the additive and the multiplicative demand cases. Table 3 summarizes the results and indicates whether each was obtained analytically (A) or experimentally (E). The analytical results are derived in this section. The experimental results are determined by implementing the algorithm presented in Section 4 to a computational study described in Section 5. In Section 7, we offer some intuition to help reconcile the somewhat conflicting results arising from the analysis of the additive and the multiplicative demand cases.

In the additive demand case, analytical results reveal that, for a fixed $T$, the optimal selling price increases over time. An analogous statement cannot be made for the optimal stocking quantity because numerical examples indicate that the optimal stocking quantity either strictly increases or first increases and then decreases over time. However, we prove that both the first-period optimal selling price and the first-period optimal stocking quantity increase as $T$ increases.

The results for the multiplicative demand case are complementary. For a fixed $T$, computational experiments suggest that the optimal selling price decreases over time, although we are unable to provide a proof. However, it can be shown that an implication of an optimal price that decreases over time is an optimal stocking quantity that increases over time. As $T$ increases, we prove that the first-period optimal selling price increases and that the corresponding optimal stocking quantity either increases or first increases and then decreases. Thus, it seems that the structure
arising in the additive demand case for a fixed $T$ provides a direct analog of the structure arising in the multiplicative demand case for a varying $T$. Similarly, additive demand results for a varying $T$ map symmetrically onto multiplicative demand results for a fixed $T$.

We begin the analysis by identifying some useful properties regarding the functions $p_{zx}$ and $q_{zx}$ and follow with some implications of those properties, given that $z_t > z_{t-1}$. Then, we focus on characterizing $z_t$, which includes verifying that $z_t > z_{t-1}$. The following two lemmas stem primarily from the analysis of a single-period problem.

**LEMMA 1:** For the additive demand case, given $z > x$,

(a) $p_{zx} = p^0(c) + \left[\mu(x) - \Theta(z, x)\right]/2b$,  
(b) $\frac{\partial p_{zx}}{\partial z} = \left[1 - G(z, x)\right]/2b > 0$,  
(c) $\frac{\partial^2 p_{zx}}{\partial z^2} = -g(z, x)/2b < 0$,  
(d) $q_{zx}$ is increasing in $z$.

**LEMMA 2:** For the multiplicative demand case, define

$$
\gamma = \begin{cases} 
1/2 & \text{if } y(p) = a - bp, \\
1 & \text{if } y(p) = ae^{-bp}, \\
b/(b-1) & \text{if } y(p) = ap^{-b}. 
\end{cases}
$$

Then, given $z > x$,

(a) $p_{zx} = p^0(c) + \gamma(c + h)\frac{\Lambda(z, x)}{\mu(x) - \Theta(z, x)} > p^0(c) > c$,  
(b) $p_{zx} = p^0(-h) + \gamma(c + h)\frac{\Lambda(z, x)}{\mu(x) - \Theta(z, x)} > p^0(-h)$,  
(c) $\frac{\partial p_{zx}}{\partial z} = \gamma(c + h)\frac{\Lambda(z, x) - \Theta(z, x)}{(\mu(x) - \Theta(z, x))^2} > 0$,  
(d) $q_{zx}$ either is increasing or is increasing, then decreasing in $z$ for a given $x$.

Lemmas 1 and 2 demonstrate basic, although symmetric, differences between the additive and multiplicative demand cases. In both cases, $p_{zx}$ is increasing in $z$; however, in the additive case, it is concave in $z$ and increasing in $x$, while in the multiplicative case, it is convex in $z$ and decreasing in $x$. Figure 1 demonstrates this reflective symmetry. Another symmetric difference occurs in the relationship between $p_{zx}$ and the benchmark coined the riskless price by Mills [23]. Riskless price refers to the price that maximizes $(p - c)D(p, \mu(x))$, the single-period profit function for the certainty-equivalent problem in which $c$ is replaced by its mean. In the additive demand case, the riskless price is state-dependent, equal to $p^0(c) + \mu(x)/2b$; thus, $p_{zx}$ is less than the riskless price in this case (Mills [23]). However, in the multiplicative demand case, the riskless price is state-independent, equal to $p^0(c)$; thus, $p_{zx}$ is greater than the riskless price in this case (Karlin and Carr [17]).

Theorem 1 below establishes that $z_t > z_{t-1}$. Thus, an implication of Lemma 1, parts (b) and (c), is that the optimal selling price in the additive demand case is increasing over time as long as demand is not observed. Once demand is observed, the selling price is set equal to $p(c)$ in accordance with having full information, and it remains constant for the remainder of the problem horizon. It is difficult in this case to ascertain how the optimal stocking quantity, $q_t = y(p_t) + z_t$, behaves over time prior to when demand first is observed because $p_t$ is increasing in $t$, which
implies that \( y(p_t) \) is decreasing in \( t \), and \( z_t \) is increasing in \( t \). However, examples indicate that \( q_t \) is unimodal in \( t \), first increasing and then decreasing (Petruzzi [26]).

Similarly, it is difficult to determine how the optimal selling price in the multiplicative demand case behaves over time as long as demand is not observed because \( p_{zx} \) increases with \( z \), but decreases with \( x \). However, we conjecture that \( p_t \) is decreasing over time. Our rationale follows.

Given Lemma 2, we can interpret the optimal selling price as including a premium over a base price, where \( p_0(c) \) denotes the base price and \( \gamma(c + h) \Lambda(z, x)/[\mu(x) - \Theta(z, x)] \) denotes the premium. Multiplying both the numerator and denominator of the premium by \( y(p_{zx}) \) indicates that the role of the premium is to recoup, on a per-sale basis, the total expected cost resulting from the management of buffer inventory, as measured by the expected cost of leftovers. The "recouping factor" is given by \( \gamma \), which is a weighting that is specific to the demand function and is related to its price elasticity. Since the premium, in effect, transfers to the customer the burden of managing the risk of uncertainty, it seems plausible that the premium should decrease over time as the firm hones in on the value of \( \epsilon \). Accepting this hypothesis, which is supported by the numerical examples that we studied, leads to the conclusion that \( p_t \) is decreasing over time, which in turn implies that \( q_t = y(p_t)z_t \) is increasing over time.

To establish Theorem 1 and further characterize the behavior of \( z_t \), we define

\[
Q_t(z, x) = \Pi(z, x) + \alpha \left[ \int_x^z V_{t+1}(u)g(u, x) \, du + \int_z^\infty W_{t+1}(z)g(u, x) \, du \right].
\]

Consequently, from (7) and the definition of \( z_t \),

\[
W_t(x) = \max_{z \geq x} Q_t(z, x) = Q_t(z_t, x)
\]

and

\[
\frac{\partial Q_t(z, x)}{\partial z} = \frac{\partial \Pi(z, x)}{\partial z}
+ \alpha \left\{ \left[ V_{t+1}(z) - Q_{t+1}(z_{t+1}, z) \right]g(z, x) + \frac{dQ_{t+1}(z_{t+1}, z)}{dz} \right\} \left[ 1 - G(z, x) \right],
\]

where \( \partial \Pi(z, x)/\partial z = (\partial \Pi(z, p, x)/\partial z)_{p=p_{zx}} \) since \( (\partial \Pi(z, p, x)/\partial p)_{p=p_{zx}} = 0 \). If we assume that \( z_{t+1} \) is an interior point, then \( z_{t+1} = z_{t+1}(z) \), where \( z_{t+1}(z) \) satisfies \( \partial Q_{t+1}(Z, z)/\partial Z = \)
Table 4. Relationships.

\[
\begin{align*}
  z - \Lambda(z, x) &= \mu(x) - \Theta(z, x) > 0 \\
  \frac{\partial g(u, x)}{\partial z} &= g(u, x) r(x) \\
  \frac{\partial [1 - G(u, x)]}{\partial z} &= -\frac{\partial G(u, x)}{\partial z} = [1 - G(u, x)] r(x) \\
  \frac{\partial \mu(x)}{\partial z} &= \mu(x) - x r(x) \\
  \frac{\partial \Theta(z, x)}{\partial z} &= \Theta(z, x) r(x) \\
  \frac{\partial \Omega(z, x)}{\partial z} &= \frac{\partial \Omega(z, p, x)}{\partial z} = \frac{p_{t+1}}{\partial p_{t+1}} = M(p_{t+1})((p_{t+1} + h)[\mu(x) - x - \Theta(z, x)] r(x) \\
  \frac{\partial \Omega(z, x)}{\partial z} &= \frac{\partial \Omega(z, p, x)}{\partial z} = \frac{p_{t+1}}{\partial p_{t+1}} = M(p_{t+1})((p_{t+1} + h)[1 - G(z, x)] - (c + h))
\end{align*}
\]

0 for a given \( z \). In such a case, \( dQ_{t+1}(z_{t+1}, z)/dz = \partial Q_{t+1}(z_{t+1}, z)/\partial z \) and, applying the applicable relationships from Table 4, (8) simplifies to

\[
\frac{\partial Q_t(z, x)}{\partial z} = M(p_{t+1})((p_{t+1} + h)[1 - G(z, x)] - (c + h)) + \alpha g(z, x) K(z_{t+1}, z), \tag{9}
\]

where

\[
K(z_{t+1}, z) = (p(z) - c) D(p(z), z) - (p_{t+1} - c) D(p_{t+1}, z) + M(p_{t+1})(c + h)(z_{t+1} - z). \tag{10}
\]

If \( z_{t+1} \) is not an interior point, then there are two possibilities, given that \( z \) is the lower bound for \( \epsilon \) as of the beginning of period \( t + 1 \): Either \( \epsilon \) also is bounded from above by some value \( B \) and \( z_{t+1} = B \), or \( z_{t+1} = z \). If \( z_{t+1} = B \), then the dynamic nature of the problem ends at the conclusion of period \( t + 1 \) and \( Q_{t+1}(z_{t+1}, z) = II(B, z; \alpha \int_0^B \frac{V_t(u)(g(u, z)}{\partial z} du \). Consequently, if \( z_{t+1} = B \), (8) becomes \( \partial Q_t(z, x)/\partial z = M(p_{t+1})((p_{t+1} + h)[1 - G(z, x)] - (c + h)) + \alpha g(z, x) K(B, z) \). Thus, (9) remains a valid expression for this case. We now demonstrate with Theorem 1 that \( z_{t+1} = z \) cannot occur.

**THEOREM 1:** Let \( x \) denote the lower bound for \( \epsilon \) as of the beginning of period \( t \). Then \( z_t > x \).

An immediate implication of Theorem 1 is that \( z_t > z_{t-1} \) for all \( t \). Another is that the expression for \( \partial Q_t(z, x)/\partial z \) given by (9) is valid for all \( t \). We use these results with the following lemma as the basis for proving Theorem 2, which characterizes the effect of a varying \( T \).

**LEMMA 3:** For \( z > x \), the function \( K(z, x) \) is strictly positive and strictly increasing in \( z \).

**THEOREM 2:** Let \( z_{t, T} \) denote the optimal value of \( z \) in period \( t \) of a \( T \)-period problem given that \( A \) is the initial lower bound for \( \epsilon \). Then \( z_{t+1, T} > z_{t, T} \).

From Theorem 2 and Lemmas 1 and 2, parts (b) and (d), it follows that: (1) The first-period optimal selling price is increasing in \( T \) both in the additive and the multiplicative demand cases and (2) the first-period optimal stocking quantity is increasing in \( T \) in the additive demand case, but it is either increasing or first increasing and then decreasing in \( T \) in the multiplicative demand case. These implications extend and generalize a result of Krouse and Senchack [18], who determined that the optimal stocking quantity for a firm that exploits information obtained from observing stock-outs is no less than the optimal stocking quantity for a firm that does not exploit such information. In their model, Krouse and Senchack [18] assume that price is exogenous and that \( F(\cdot) \) represents an exponential distribution.
4. SOLVING THE DYNAMIC PROGRAM

In this section, we develop a solution procedure by establishing a monotone dependence between \( z_t \) and \( z_T \). Consequently, we can choose an arbitrary candidate for the last period decision for \( z \) and demonstrate that if the candidate is in fact the optimal choice, then a corresponding value for \( z_{T-1} \) is implied. Continuing this recursion, we can determine an imputed value for the initial lower bound on \( \epsilon \), which we call \( z_0 \). We can test the candidate solution simply by comparing \( z_0 \) to \( A \). If \( z_0 = A \), then the candidate choice for \( z_T \) and the corresponding values computed for \( z_t \) \((t = T - 1, \ldots, 1)\) indeed may be optimal and are saved. If \( z_0 \neq A \), or, because of Theorem 1, if \( z_t \leq A \) for any \( t \geq 1 \), then the candidate solution is not optimal and is discarded. Once all candidates are tested, \( W_1(A) \) can be computed for each of the saved solutions to determine the answer. Theorem 3 provides the justification for the procedure.

**THEOREM 3:** Given \( z_{t+1} \) and \( z (z < z_{t+1}) \), \( \partial Q_t(z, x) / \partial z = 0 \) can be solved uniquely for \( x (x < z) \).

Using Theorem 3, the complexity of the search for the optimal solution to the dynamic program (7) in \( T \) variables can be reduced to a search for just one variable. Adapting an argument from Dada and Srikanth [9], suppose that \( z_t \) and \( z_{t+1} \) are known. Then Theorem 3 implies that solving \( \partial Q_t(z, x) / \partial z \big|_{z=z_t} = 0 \) for \( x \) yields a unique solution. In an optimal solution, if \( z_t \) is an interior point, then this imputed value for \( x \) represents \( z_{t-1} \) and, because of Theorem 1, it also is an interior point. Consequently, it satisfies \( \partial Q_{t-1}(z, x) / \partial z \big|_{z=z_{t-1}} = 0 \), which then can be used to identify \( z_{t-2} \) by again solving for \( x \). Therefore, it follows that given \( z_T \) and working sequentially from period 1 to period 1, all remaining values \( \{z_1, \ldots, z_{T-1}\} \) can be recovered. Hence, solving the dynamic program (7) reduces to a line search for \( z_T \), whose value can be found using the following algorithm:

**Algorithm.** Given \( T \), define \( Z_{T+1} = 0 \) and specify a tolerance level \( \delta > 0 \)

Define \( z_0(A) \) as the solution to the problem in which \( T = 1 \)

FOR \( z_0(A) \leq z_T \), DO

Systematically propose a value for \( z_T \); call this value \( Z_T \)

FOR \( t = T, \ldots, 1 \), DO

Given \( z_{t+1} = Z_{t+1} \) and \( z = Z_t \), solve \( \partial Q_t(z, x) / \partial z = 0 \) for \( x \)

IF solution exists, THEN let \( Z_{t-1} = x \)

ELSE trial vector \( \{Z_T, \ldots, Z_t\} \) not optimal: discard

END FOR LOOP

IF computed value for \( Z_0 \) is such that \( |Z_0 - A| < \delta \), THEN save trial

vector \( \{Z_T, \ldots, Z_1\} \) as candidate solution vector

ELSE trial vector \( \{Z_T, \ldots, Z_1\} \) is not optimal: discard

END FOR LOOP

Of the candidate solution vectors, select as optimal the one that yields the maximum

for \( W_1(A) \)

In summary, assuming that \( z_T \) is an interior point, Theorem 3 guarantees that all intermediate decisions \( \{z_1, \ldots, z_{T-1}\} \) can be recovered analytically from \( z_T \). Consequently, we guess a value for \( z_T \), compute the vector \( \{z_1, \ldots, z_{T-1}\} \) corresponding to the guess, and then test the optimality of the guess first by comparing to \( A \) the imputed value for \( z_0 \) associated with the guess and then, if necessary, calculating the value of the profit function associated with the guess. If one of the intermediate decisions, \( z_t \), cannot be recovered from a particular guess, then it means that the imputed value for \( z_{t+1} \) cannot solve the equation \( \partial Q_{t+1}(z, x) / \partial z \big|_{z=z_{t+1}} = 0 \), implying that
and $Y_z$ cannot be an interior point. But Theorem 1 ensures that if $z_T$ is an interior point, then all $z_i$ are interior points. Thus, if a particular guess for $z_T$ is chosen as an interior point and the intermediate decision $z_i$ cannot be recovered, then the contradiction implies that the guess is not optimal and can be eliminated. Also, since Theorem 1 implies that $z_{1,T} > z_1$ and Theorem 2 implies that $z_{1,T} > z_{1,1} \equiv z_{s}(A)$, we can bound the range over which we need to guess values for $z_T$. Once all guesses in this restricted range are exhausted, the procedure ends. We note that although it might be possible for more than one guess to pass the first test (i.e., yield a $z_0 = A$), experimentation with the algorithm has not generated an instance in which such a case occurs.

Next, we discuss the possibility of a boundary-point solution.

If $B$ exists as a finite upper bound for $e$, then $T$ different boundary points also must be tested as possible solutions to the dynamic program. Thus, there are a total of $T + 1$ candidate vectors, including the possibility of an interior point solution. These are itemized in Table 5. Option 1 is a degenerate candidate and we can generate the other $T$ candidates as follows: For Option $i$, set $z_i = B$ and implement the algorithm using $i – 1$ in place of $T$. Given the $T + 1$ options, we can perform a preliminary test for the optimality of Option $i$ by checking the sign of $\partial Q_i(z, x)/\partial z$ when evaluated at $z = B$ and $x = z_{i-1}$. If the sign is negative, then the option violates the Kuhn–Tucker optimality conditions and can be discarded. To choose the best among all options that pass the preliminary test, we need to evaluate $W_1(A)$ for each option.

### 5. ILLUSTRATIONS

In this section, we illustrate the mechanics of the algorithm by applying it to a special case of the additive demand model in which $\epsilon$ is characterized initially by an exponential distribution. Then, we describe the computational study in which the algorithm was implemented to generate the experimental results used to complement the analytical results developed in Section 3.

**ILLUSTRATIVE EXAMPLE OF ALGORITHM:** For the purpose of this example, we let $D(p, c) = a – bp + \epsilon$ and $1 – F(u) = e^{-\lambda u} (u \geq A = 0)$. Correspondingly, $M(p_{zx}) = 1, \mu(A) = \mu(0) = 1/\lambda, 1 – G(u, x) = e^{-\lambda(u-x)}$, and $\mu(x) = \mu(A) + x$. In addition, to simplify the recursion formulas, we define the following: $\kappa_1 \equiv 2b(c + h), \kappa_2 \equiv a – bc + \mu(A), Y_t \equiv 1 – F(z_t)$, and $Y_x \equiv 1 – F(x)$.

Given these specifications and definitions, we can write:

- \( p(x) = \arg \max_{p} \{(p – c)D(p, x)\} = (a + x + bc)/2b, \)
- \( p_{zx} = (a + bc)/2b + [\mu(x) - \Theta(z, x)]/2b = (a + x + bc)/2b + [\Theta(x, x) - \Theta(z, x)]/2b, \)
- \( \Theta(z, x) = \int_{z}^{\infty} (u-z)g(u, x)du = \int_{z}^{\infty} (u-z)e^{-\lambda(u-x)} du = \mu(A)e^{-\lambda(z-x)}. \)

Thus

- \( [p_{zx} + h][1 – G(z_t, x)] = (c + h) = \frac{1}{2b} \left\{ \left[ \kappa_1 + \kappa_2 + x - \mu(A) \frac{Y_t}{Y_x} \right] \frac{Y_t}{Y_x} - \kappa_1 \right\}, \)
• $K(z_{t+1}, z_t) = [p(z_t) - c]D(p(z_t), z_t) - [p_{z_{t+1}} - c]D(p_{z_{t+1}}, z_t) + (c + h)(z_{t+1} - z_t) = \frac{\mu(A)}{2b} \left\{ \kappa_1 \ln \frac{Y_t}{Y_{t+1}} + \frac{\mu(A)}{2} \left[ 1 - \frac{Y_{t+1}}{Y_t} \right]^2 \right\}.$

Hence, the condition $[\partial Q_t(z, x)/\partial z]_{z=z_t} = [p_{z_{t+1}} + h][1 - G(z_t, x)] - (c + h) + \alpha g(z_t, x)K(z_{t+1}, z_t) = 0$ reduces to

$$\kappa_1 Y_x^2 - [(\kappa_1 + \kappa_2 + J_t)Y_x] + [\mu(A)Y_x] \ln Y_x + \mu(A)Y_x^2 = 0,$$

(11)

where

$$J_t = \alpha \kappa_1 \ln \frac{Y_t}{Y_{t+1}} + \frac{\mu(A)}{2} \left[ 1 - \frac{Y_{t+1}}{Y_t} \right]^2.$$

(12)

Notice that (11) and (12) are not specified in terms of the $z_t$’s for which we are seeking values. Instead, they are specified in terms of $Y_t$’s, which represent the cumulative distribution functions associated with the $z_t$’s. This is convenient for two reasons. First, for this example, the distribution function is invertible and thus, the desired $z_t$-values can be obtained easily from derived $Y_t$-values (in particular, $z_t = -\mu(A) \ln Y_t$). And second, an exhaustive search for $Y_T$ requires less effort than an exhaustive search for $y_T$ because $Y_T$ is bounded, but $y_T$ is not.

Given (11) and (12), the algorithm applies as follows: By definition, $J_T = 0$. Thus, given a proposed value for $Y_T, Y_{T-1}$ is determined by solving (11) for $Y_x$ [recall that Theorem 3 guarantees that (11) has a unique solution for $Y_x$ in the applicable range]. Then, given $Y_T$ and $Y_{T-1}, j_{T-1}$ is computed from (12). This, in turn, is used with $Y_{T-1}$ in (11) to solve for $Y_x$, which provides the value for $Y_{T-2}$. This procedure continues until a value for $Y_0$ is determined. If the derived value of $Y_0$ is equal to 1 (thereby corresponding to a derived value for $Z_0$ equal to A), then the proposed value of $Y_T$ and the computed values of $(Y_{T-1}, \ldots, Y_1)$ provide a candidate solution and hence, are saved. Otherwise, $Y_T$ and $(Y_{T-1}, \ldots, Y_1)$ are discarded. Regardless, a new value for $Y_T$ is systematically proposed and the procedure is repeated.

To demonstrate this procedure, we set $T = 3, \lambda = 1/210, c = 10, h = 1, a = 840, b = 21$, and $\alpha = 0.9$. Correspondingly, $\mu(A) = 210, \kappa_1 = 462$, and $\kappa_2 = 840$. The results of the search procedure using these parameters are recorded in Table 6.

Table 6 indicates that the solution procedure returns only one candidate solution for this example, namely, $(Y_1, Y_2, Y_3) = (0.2346, 0.0491, 0.0121)$. Accordingly, this solution represents the global optimum. Correspondingly, the optimal first-period decision for this 3-period example is $z_1 = -\mu(A) \ln Y_1 = 304.5$. Notice that, for this example, we restricted our search to proposed values of $Y_3$ no greater than $Y_3 = 0.3779$ (i.e., to proposed values of $z_3 \geq 204.3$). This is because $Y_3(z) = 0.3779$ (i.e., $z_3(A) = 204.3$) represents the solution to the single-period problem (i.e., $z_3(A) = 204.3$ solves $[p_{z_A} + h][1 - F(z)] = (c + h)$).

DESCRIPTION OF COMPUTATIONAL STUDY: To generate the experimental results reported in Table 3, we implemented the algorithm for a variety of problem instances. In this subsection, we provide an overview of this computational study; further details can be found in Petruzzi [26].

We solved a series of $T$-period problems ($T = 1, 2, 3, 4, 5$) each for the additive demand case, using $y(p) = a - bp$, and for the multiplicative demand case, using $y(p) = ae^{-bp}$. In doing so, we arbitrarily chose $\alpha = 0.9$ and $c = 10$; but we considered two values for $h$ based on $c (h = 0.1c = 1$ and $h = 0.2c = 2)$ and we specified two alternative forms for $F(\cdot)$: $F(x) = (x - A)/(B - A)$
for $A \leq x \leq B$ (uniform distribution) and $F(x) = 1 - e^{-\lambda(x-A)}$ for $x \geq A$ (shifted-exponential distribution).

Rather than stipulate values for $a$, $b$, $A$, and $B$ directly for each of the cases considered, we stipulate values for more intuitive measures of the demand function and then derived values for $a$, $b$, $A$, and $B$ to correspond to those measures. The measures are based on the demand function when evaluated at the period-1 riskless price, which, recall, is the price that maximizes $(p-c)D(p_r, \mu(A))$. The measures are the price-elasticity of period-1 expected demand and the coefficient of variation of period-1 demand ($p_r$ is used to denote period-1 riskless price):

- $\beta = -\left(\frac{\partial E[D(p_r, \mu(A))]}{\partial p_r}\right)_{p_r} = \text{price-elasticity},$
- $CV = \sqrt{\frac{\text{Var}[D(p_r, \mu(A))]}{E[D(p_r, \mu(A))]}^{1/2}} = \text{coefficient of variation}.$

For the additive demand case, we fix $A = 0$ and $[a + \mu(A)] = 1050$ and note that $p_r = [a + \mu(a) + be]/2b$. Thus, $\beta = [a + \mu(A) + be]/[a + \mu(A) - be] = [1050 + 10b]/[1050 - 10b]$ and $CV = 2\sigma(A)/[a + \mu(A) - be] = 2\sigma(A)/[1050 - 10b]$, where $\sigma(A)$ denotes the period-1 standard deviation of $\epsilon$. Consequently, if $F(u)$ corresponds to a uniform distribution, then we can write: $b = 105(\beta -1)/[\beta +1], B = (12)^{1/2}\sigma(A) = [(2100)(3)^{1/2}]CV/(\beta + 1)$, and $a = (a + \mu(A)) - \mu(A) = 1050 - B/2$. And, if $F(u)$ corresponds to an exponential distribution, then we can write: $b = 105(\beta -1)/[\beta +1] + a = (a + \mu(A)) - \mu(A) = 1050 - \sigma(A) = 1050[1 - CV/(\beta + 1)]$. Thus, to complete a specification of a problem instance for the additive demand case, it suffices to specify a $(\beta, CV)$ pair. We experiment with nine such pairs by testing three values each for $\beta$ and $CV$: $\beta = 1.5, 2.0, 2.5$; and $CV = 0.1, 0.3, 0.5$.

For the multiplicative demand case, we fix $a = 1$ and note that $p_r = [1 + bc]/b$. Thus, $\beta = 1 + bc = 1 + 10b$ and $CV = \sigma(A)/\mu(A)$. Moreover, we set $\mu(A) = 1050e^\beta/(\beta + 1)$, which

<table>
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<th>Proposed $Y_3$</th>
<th>Derived $Y_2$</th>
<th>Computed $J_2$</th>
<th>Derived $Y_1$</th>
<th>Computed $J_1$</th>
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</table>
provides a form of normalization because it ensures that, for a given value of $\beta$, period-1 expected demand, when evaluated at the period-1 riskless price, is equivalent for both the additive and the multiplicative demand cases. Consequently, if $F(u)$ corresponds to a uniform distribution, then we can write: $b = (\beta - 1)/10, B = \mu(A) + (3^{1/2})\sigma(A) = \mu(A)[1 + (3^{1/2})CV]$, and $A = \mu(A)[1 - (3^{1/2})CV]$. And, if $F(u)$ corresponds to a shifted-exponential distribution, then we can write: $b = (\beta - 1)/10$ and $A = \mu(A) - \sigma(A) = \mu(A)(1 - CV)$. Thus, as in the additive demand case, to complete a specification of a problem instance for the multiplicative demand case, it suffices to specify a $(\beta, CV)$ pair. We test the same nine $(\beta, CV)$ pairs as in the additive demand case.

To summarize the design of our computational study, we fix $A = 0$ and $[a + \mu(A)] = 1050$ for the additive demand case; but, for the multiplicative demand case, we instead fix $a = 1$ and set $\mu(A) = 1050e^{\beta}/(\beta + 1)$. Then, for both the additive and the multiplicative demand cases, we (1) specify $a, b, A$, and $B$ in terms of $\beta$ and $CV$, (2) fix $a = 0.9$ and $c = 10$, and (3) vary the remaining parameters as follows: $h = 1, 2; \beta = 1.5, 2.0, 2.5; CV = 0.1, 0.3, 0.5; T = 1, 2, 3, 4, 5; and F(u) = uniform, shifted-exponential$.

Rather than reproduce the results of this study here, we refer to Table 3 (Section 3). However, Figures 2 and 3 below, which are representative of the computational results, provide additional insight. In these figures, we plot $F(z_{t,T})$, which represents the probability that learning is completed in or before period $t$ of a $T$-period problem. Reading the graphs vertically along one of its columns indicates that for a given $T$, the probability that learning is completed within 3 or 4 periods quickly approaches 1. Also, reading the graphs horizontally across columns indicates that the increase of periods to the problem horizon has little effect on the first period optimal decisions once $T$ is greater than 3 or 4 periods. This has implications for applying our model to more realistic management scenarios because it suggests that richer $T$-period learning problems can be approximated by a 2- or 3-period rolling horizon dynamic learning model.

6. GENERALIZATION TO INCLUDE CASES OF A NON-PERISHABLE PRODUCT

Although formulated in the context of a perishable product, the dynamic program (4) can be extended in a variety of ways. The ensuing tractability depends on the degree to which the resulting program can be reduced to analogs of (7), which exhibits appropriate separability. We demonstrate one such extension by generalizing (4) to include cases of a non-perishable product.
For this extension, we slightly modify the notation, reinterpreting $h$ as an inventory carrying cost ($h < c$) and replacing $V_t(u)$ with the following definition:

$$V_t(u, I) = \text{maximum discounted profit from period } t \text{ to } T \text{ under full information,}$$
$$\text{given that it is known that } \epsilon = u \text{ and that } I \text{ is the initial inventory on hand.}$$

Then, generalizing (4),

$$W_t(x) = \max_{z, p} \left\{ \Pi(z, p, x) + \alpha \left[ \int_x^z V_{t+1}(u, \rho(Leftovers(z, p, u))) g(u, x) du \right. \right.$$  
$$\left. + \int_x^\infty W_{t+1}(z) g(u, x) du \right\}, \quad (13)$$

where $\rho(\cdot)$ is a function that translates the number of leftovers from period $t$ into a beginning inventory for period $t+1$. Thus, this formulation includes the perishable-product case [$\rho(X) = 0$], along with other cases of obsolescence or disposal [$0 < \rho(X) < X$], as well as the case of no attrition [$\rho(X) = X$]. Correspondingly, we specify that $\rho(0) = 0$, $\rho(X) \leq X$, and $0 \leq d\rho(X)/dX \leq 1$. The resulting separability of (13) depends on the form of $Leftovers(z, p, u)$.

In the multiplicative demand case, $Leftovers(z, p, u) = y(p)(z - u)$, which is a function that depends both on $z$ and $p$. Consequently, subsequent periods are linked by the pricing decision as well as the decision for $z$. This results in the loss of the separability property, making the multiplicative demand case of this extension particularly challenging to analyze. We identify this challenge as a viable direction of research and focus instead on the additive demand case, which, it turns out, remains completely tractable within the analysis framework already presented.

In the additive demand case, $Leftovers(z, p, u) = z - u$, which is independent of $p$. Consequently, subsequent periods are linked only by the decision for $z$, as was the case in (4). Correspondingly, Lemma 1 applies and (13) reduces to the analog of (7). This leads to the following theorem.

**THEOREM 4:** For the additive demand case, if $0 \leq \partial V_t(u, I)/\partial I \leq c$, then Theorems 1–3 apply to the problem formulation given by (13).
Theorem 4 provides a sufficient condition for which all of the results from Sections 3 and 4 hold for this more general formulation of the additive demand problem. However, we argue that this sufficient condition must hold in most practical circumstances. To demonstrate, first assume that $\partial V_t(u, I)/\partial I > c$ for some $I$. This suggests that profit could be improved simply by ordering additional stock immediately at the start of period $t$ and beginning the $(T-t+1)$-period deterministic problem with $> I$ units as the initial inventory on hand rather than beginning with $I$. Of course, this results in a contradiction of $V_t(u, I)$ as the value of the optimal profit, which implies that the assumption that $\partial V_t(u, I)/\partial I > c$ is incorrect. Establishing the lower bound is slightly more restrictive, but if we allow for free disposal, then the argument is similar. Assume that $\partial V_t(u, I)/\partial I < 0$ for some $I$. Then, an improvement is possible simply by disposing units and beginning with $< I$ rather than beginning with $I$. Again, this contradicts the definition of $V_t(u, I)$, which implies that $\partial V_t(u, I)/\partial I \geq 0$.

Therefore, we find that for the additive demand case, the assumption of a perishable product is not necessary and that the results developed in this paper apply to a much more general modeling framework. Notice that we need not actually specify $V_t(u, I)$, but instead require only a property of the optimal solution with which it is associated, namely, that $0 \leq \partial V_t(u, I)/\partial I \leq c$. For one particular specification of $V_t(u, I)$ involving a case in which free disposal is allowed, we refer to Petruzzi [26]. In the case considered there, the sufficient condition of Theorem 4 is satisfied. Finally, we note that we could generalize (13) further by incorporating a fixed cost of purchasing. The effect of such a cost is minimal because its impact on the model is relevant only after leftovers occur for the first time; otherwise a purchase order is required regardless. Thus, in terms of developing the optimal policy during uncertainty, we can assume that any fixed purchasing cost is absorbed into $V_t(u, I)$, implying that Theorem 4 applies as stated. A modified version of the Wagner–Whitin-type algorithm developed by Thomas [33] will yield the optimal policy associated with the deterministic subproblem characterized by $V_t(u, I)$ when a fixed cost of purchasing is considered. Gallego and van Ryzin [12] investigate related, continuous-time models. Other nonlinear cost functions also could be incorporated into (13), but such extensions likely would present challenges similar to those encountered in the multiplicative version of this extension.

7. CONCLUDING COMMENTS

The newsvendor problem has been studied extensively (see Porteus [29] for an authoritative review). However, extensions that jointly examine the modeling of pricing and learning have received relatively little attention. In this paper we present a dynamic model that simultaneously links price, learning from sales, and inventory. For the case of a perishable product, we show that the structure of the optimal policy depends crucially on whether uncertainty has an additive or a multiplicative effect on demand. For example, in the additive case, the optimal price rises until uncertainty is resolved, while in the multiplicative case, the optimal price likely falls until uncertainty is resolved. We next offer some intuition for these intriguingly different results.

Given the construct of our model, namely that demand is a deterministic but initially unknown, function of price, when a stockout occurs, the event is indicative of a fixed gap between demand and stocking quantity. By fixed we mean that if neither the stocking quantity nor the selling price is adjusted following a period in which stockouts occur, then the exact same number of shortages will result in the subsequent period. Consequently, it is natural that the firm will take action to reduce such a gap. This explains why the surrogate decision $z$ increases over time: From (3), a higher $z$ provides a tighter gap because a higher $z$ represents an increase in stocking quantity relative to demand. However, given that the objective is to reduce this gap, there are multiple
methods by which the firm can achieve the result. One such method is to increase stocking quantity while leaving selling price unadjusted; another is to increase selling price while leaving stocking quantity unadjusted; and yet another is to increase both.

To settle on a specific alternative for closing the gap between demand and stocking quantity, it is useful to take a broader view. Since the firm needs to adjust at least one of its decision variables, there is good reason for the firm to adjust its price. The reason is that the uncertainty remaining in demand is a function of price, but it is not a function of quantity. Consider that in the additive demand case, the variance of demand is not price-dependent, but the coefficient of variation is:

$$CV(Demand) = \frac{VAR(\epsilon)^{1/2}}{y(p) + E[\epsilon]}.$$ Thus, in the additive demand case, a price increase is prudent because it contributes to reducing not only the size of the gap, but also the amount of uncertainty (as measured by the $CV$) that will continue to surround the demand function if a stockout again occurs in the subsequent period. Depending on the size of the price increase and the risk associated with overcompensating for the size of the gap, even a decrease in stocking quantity could make sense as long as the magnitude of any decrease in stocking quantity remains less than the magnitude of the decrease in demand that will result from the price increase.

In the multiplicative demand case, the demand coefficient of variation does not depend on price, but the variance does:

$$VAR(Demand) = y(p)^2VAR(\epsilon).$$ Thus, in the multiplicative demand case, a price decrease is prudent because it contributes to reducing the amount of uncertainty (as measured by the $VAR$) that will continue to surround the demand function if a stockout again occurs in the subsequent period. However, since a price decrease will have an effect that is opposite to the desired result of decreasing the gap between demand and stocking quantity, a corresponding increase in stocking quantity is required. Moreover, the increase in stocking quantity must more than offset the increase in demand resulting in the price decrease to close the gap.

The generalization of the additive model to include cases in which the product is not perishable yields analogous structure to the case of a perishable product; however, a corresponding generalization of the multiplicative model yields new challenges that we identify as a direction for continued research.

As another important direction for research, we propose the more realistic extension in which inherent randomness is incorporated in $\epsilon$ such that the probabilistic nature of the problem does not end at the first occurrence of leftovers. In such a scenario, difficulties arise because of the more complicated nature with which the future is linked to the present. With inherent randomness, $z_t$ no longer represents a sufficient statistic that captures the historical information required to derive the updated probability distribution for period $t + 1$. Instead, a complete history of the $z$ decisions likely is required, which makes the updating procedure cumbersome and consequently, the embedded optimization problem particularly challenging to analyze.

However, the insight and solution procedure developed in this paper offers the following approach to addressing this challenge: Figures 2 and 3 (Section 5) indicate that only the first two or three periods of a $T$-period horizon have a material effect on current decisions. This suggests the viability of a heuristic that is based on appending the model developed in this paper to a complementary model developed by Petruzzi and Dada [28], who analyze a 2-period problem in which demand is stochastic and sales information is used to update the demand distribution between periods. In particular, given a $T$-period problem in which demand includes a stochastic term, the model developed here could be applied to periods 2 through $T$, thereby collapsing the total discounted profit associated with those periods to the equivalent of a single-period effect. Then, a corresponding 2-period problem that explicitly incorporates the stochastic nature of demand could be solved using the approach from Petruzzi and Dada [28]. We conjecture that this methodology would provide a suitable approximation to the original $T$-period problem in which price-dependent demand is stochastic and censored-information is used to revise forecasts.
Our model lays the foundation for addressing these more complicated modeling alternatives and provides insight into the tradeoffs involved when selling prices and stocking quantities are set jointly. The results raise interesting managerial issues. They indicate that when managers are uncertain of the market potential, but can learn more by observing the reaction of the market to their decisions, it is advisable to set a higher z with the implication of possibly resolving more quickly the uncertain information. Since a higher z increases the likelihood of having leftovers, this decision can be viewed as an investment today to reap higher benefit in the future. Since a higher z results in a higher price, this interpretation suggests that the investment is at least partially subsidized by the customer.

**APPENDIX: PROOFS**

**PROOF OF LEMMA 1:** Part (a). From (5) and Table 2, \( \partial \Pi(z, p, x) / \partial p = a + bc - 2bp + \mu(x) - \Theta(z, x) \), which is linearly decreasing in p and is strictly greater than zero for \( p = 0 \). Thus \( p_{xe} = (a + bc)/2b + [\mu(x) - \Theta(z, x)]/2b = p^0(c) + [\mu(x) - \Theta(z, x)]/2b \).

Parts (b) and (c) follow directly from Part (a), Table 2, and Table 4.

Part (d). From (3), (1), and Part (b): \( \partial q_{xe}/\partial z = -b p_{xe}/\partial z = [1 + G(z, x)]/2 > 0 \).

**PROOF OF LEMMA 2:** Part (a). Let \((\eta, \beta, \gamma) = (0, 0, \frac{1}{2})\) if \(y(p) = a - bp\); \((\eta, \beta, \gamma) = (1, 0, 1)\) if \(y(p) = ae^{-bp}\); and \((\eta, \beta, \gamma) = (1, 1, b/(b - 1))\) if \(y(p) = ap^{-b}\). Then, from (5), Table 1, and Table 2, we can write

\[
\frac{\partial \Pi(z, p, x)}{\partial p} = \frac{by(p)^2}{\gamma p^2} [((\mu(x) - \Theta(z, x))(p^0(c) - p) + \gamma(c + h)\Lambda(z, x)].
\]

Since \(by(p)^2/\gamma p^2 > 0\), \(\partial \Pi(z, p, x)/\partial p = 0\) if and only if \(\{\mu(x) - \Theta(z, x)\}(p^0(c) - p) + \gamma(c + h)\Lambda(z, x) = 0\).

But \(\{\mu(x) - \Theta(z, x)\}(p^0(c) - p) + \gamma(c + h)\Lambda(z, x)\) is linearly decreasing in \(p\) and is strictly greater than zero for \(p = 0\). Thus, \(p_{xe} = p^0(c) + \gamma(c + h)\Lambda(z, x)/[\mu(x) - \Theta(z, x)]\), or \(p_{xe} = p^0(c) - p = p_{xe} = \gamma(c + h)\Lambda(z, x)/[\mu(x) - \Theta(z, x)]\).

Part (b). From Part (a),

\[
\frac{\partial p_{xe}}{\partial z} = \gamma(c + h) \frac{\mu(x) - \Theta(z, x) - z [1 - G(z, x)]}{[\mu(x) - \Theta(z, x)]^2} = \gamma(c + h) \frac{\int_x^z u g(u, x) \, du}{[\mu(x) - \Theta(z, x)]^2}
\]

and

\[
\frac{\partial^2 p_{xe}}{\partial z^2} = \gamma (c + h) g(z, x) \frac{1 - G(z, x)}{[\mu(x) - \Theta(z, x)]^2} - \frac{2[1 - G(z, x)]}{\mu(x) - \Theta(z, x)} \frac{\partial p_{xe}}{\partial z} = \frac{1 - G(z, x)}{\mu(x) - \Theta(z, x)} \left[ (p_{xe} - p^0(-h)) r(z) - 2 \frac{\partial p_{xe}}{\partial z} \right].
\]

Consider:

(i) From (2), Table 2, and Part (a), \(z \rightarrow x\) implies: \(1 - G(z, x) \rightarrow 1; \mu(x) - \Theta(z, x) \rightarrow x;\) and \(p_{xe} - p^0(-h) \rightarrow \gamma(c + h) > 0\). Moreover, \(z \rightarrow x\) implies \(\partial p_{xe}/\partial z \rightarrow 0\). Therefore, \(z \rightarrow x\) implies \(\partial^2 p_{xe}/\partial z^2 \rightarrow \gamma(c + h) r(x)/x > 0\).

(ii) \(\frac{\partial^2 p_{xe}}{\partial z^2} \bigg|_{\partial^2 p_{xe}/\partial z^2 = 0} = \frac{1 - G(z, x)}{\mu(x) - \Theta(z, x)} \left[ (p_{xe} - p^0(-h)) \frac{r(z)}{x} + r(z) \frac{\partial p_{xe}}{\partial z} \right] > 0\).

From (ii), if \(\partial^2 p_{xe}/\partial z^2\) crosses the z-axis, it changes sign from negative to positive. However, from (i), \(\partial^2 p_{xe}/\partial z^2\) starts out positive. Therefore, \(\partial^2 p_{xe}/\partial z^2\) does not cross the z-axis, which implies that \(\partial^2 p_{xe}/\partial z^2 > 0\).

Part (c) follows directly from Part (a) and Table 4.

Part (d). From (3), (1), and Part (b),

\[
\frac{\partial q_{xe}}{\partial z} = \begin{cases} 
\frac{[y(p_{xe}) - bx]}{y(p_{xe})} & \text{if } y(p) = a - bp, \\
\frac{1 - b}{b} \frac{\partial p_{xe}}{\partial z} & \text{if } y(p) = ae^{-bp}, \\
\frac{[\partial q_{xe}]}{p_{xe}} & \text{if } y(p) = ap^{-b}.
\end{cases}
\]

In all three cases the following are true: (i) \(\partial q_{xe}/\partial z > 0\); (ii) \(\partial q_{xe}/\partial z = 0\) if and only if the corresponding term in \(\cdots\) equals 0; and (iii) the corresponding term in \(\cdots\) is decreasing in \(z\). Therefore, \(q_{xe}\) starts out increasing in \(z\) and changes directions at most one time.
**PROOF OF THEOREM 1:** The proof is by induction. Suppose $z_T = x$. This would imply that $\partial Q_T(z, x)/\partial z \leq 0$ when evaluated at $z = x$. But, from (9), $\partial Q_T(z, x)/\partial z = M(p_{xx})(p_{xx} - c) > 0$, which is a contradiction. Therefore, $z_T < x$. Now, assume, for the induction hypothesis, that $z_{t+1} > x$, where $z$ is the lower bound for $\epsilon$ at the beginning of $k$. This implies that either $z_{t+1} = z_{t+1}(z)$ or $z_{t+1} = B$. Therefore, $\partial Q_t(z, x)/\partial z$ is given by (9). Suppose then that $z_1 = x$. This would imply that $\partial Q_1(z, x)/\partial z \leq 0$ when evaluated at $z = x$. But, because $z_{t+1} > x$ by the induction hypothesis and $(p(z) - c)D(p(z), z) = (p_{t+1}(z) - c)D(p_{t+1}(z), z)$ by the definition of $p(z)$ as the value of $p$ that maximizes the function $(p - c)D(p, c)$, we get from (9) and (10) that $\partial Q_1(z, x)/\partial z = M(p_{xx})(p_{xx} - c) > 0$. Since this is a contradiction, we conclude that $z_1 > x$.

**PROOF OF LEMMA 3:** From (10), if $z > x$, then $K(z, x) > (p(x) - c)D(p(x), x) - (p_{xx} - c)D(p_{xx}, x)$. But $(p(x) - c)D(p(x), x) > (p_{xx} - c)D(p_{xx}, x)$ because $p(x)$ maximizes the function $(p - c)D(p, c)$, by definition. Therefore, $K(z, x) > 0$. We prove the second part of the lemma separately for the additive and the multiplicative demand cases.

For the additive demand case, $D(p, x) = y(p) + x$ and $M(p) = 1$. Thus, from (10), Lemma 1, and Table 4,

$$\frac{\partial K(z, x)}{\partial z} = [\mu(x) - \Theta(z, x) - x] \frac{\partial p_{xx}}{\partial z} + (c + h) > 0.$$ 

For the multiplicative demand case, $D(p, x) = y(p)x$ and $M(p) = y(p)$. Thus, from (10), $K(z, x) = (p(x) - c)y(p(x)x) + y(p)z + (p_{xx} - h)x$; let $L(z, x) = [(c + h)z - (p_{xx} + h)x]$; then $K(z, x)$ behaves in $z$ as $K_0(z, x) = y(p)xz$. Do this. Thus, from Lemma 2,

(a) $L(z, x)/\partial z = (c + h) - x[\partial p_{xx}/\partial z]$ and $\partial^2 L(z, x)/\partial z^2 = -x[\partial^2 p_{xx}/\partial z^2] < 0.$

(b) From (a), $x(\partial p_{xx} / \partial z) = (c + h) - \partial L(z, x)/\partial z$, 

$$\frac{\partial p_{xx}}{\partial z} = \frac{(c + h)\partial L(z, x)}{\partial x} + (p_{xx} + h) \frac{\partial L(z, x)}{\partial z}.$$ 

(c) If $y(p) = ae^{-by}$ or $y(p) = ap^{-b}$, then $dy(p)/dp = -by(p)p^\beta$, where $\beta = 0$ for $y(p) = ae^{-by}$ and $\beta = 1$ for $y(p) = ap^{-b}$. Further, from Lemma 2, $b(p_{xx} + h) - p_{xx}^\beta = b(c + h)z[\mu(x) - \Theta(z, x)]$. Thus, for these cases, from (a),

(c.1) $\frac{\partial K_0(z, x)}{\partial z} = y(p_{xx}) \frac{\partial L(z, x)}{\partial x} - \frac{b(p_{xx})}{p_{xx}^\beta} \frac{\partial p_{xx}}{\partial z} \frac{\partial L(z, x)}{\partial z} > y(p_{xx}) \frac{\partial L(z, x)}{\partial z}$

and, from (b),

(c.2) $\frac{\partial K_0(z, x)}{\partial z} > \frac{y(p_{xx})}{p_{xx}^\beta} \frac{\partial L(z, x)}{\partial x} - \frac{b(p_{xx})}{p_{xx}^\beta} \frac{\partial p_{xx}}{\partial z} \frac{\partial L(z, x)}{\partial z}$.

Therefore, if $\partial L(z, x)/\partial z > 0$, then $\partial K_0(z, x)/\partial z > 0$ by (c.1); and if $\partial L(z, x)/\partial z < 0$, then $\partial K_0(z, x)/\partial z < 0$ by (c.2). Therefore, if $y(p) = ae^{-by}$ or $y(p) = ap^{-b}$, then $\partial K_0(z, x)/\partial z > 0$, which implies that $\partial K(z, x)/\partial z > 0$.

(d) Likewise, if $y(p) = a - bp$, then $dy(p)/dp = -b$. Thus, for this case,

(d.1) $\frac{\partial K_0(z, x)}{\partial z} = y(p_{xx}) \frac{\partial L(z, x)}{\partial x} - \frac{b(p_{xx})}{p_{xx}^\beta} \frac{\partial p_{xx}}{\partial z} \frac{\partial L(z, x)}{\partial z} > y(p_{xx}) \frac{\partial L(z, x)}{\partial z}$

and, from (b),

(d.2) $\frac{\partial K_0(z, x)}{\partial z} > \frac{y(p_{xx})}{p_{xx}^\beta} \frac{\partial L(z, x)}{\partial x} - \frac{b(p_{xx})}{p_{xx}^\beta} \frac{\partial p_{xx}}{\partial z} \frac{\partial L(z, x)}{\partial z}$.

Therefore, as in the final step of part (c) of the proof, $\partial K(z, x)/\partial z > 0$ if $y(p) = a - bp$, which completes the proof.

**PROOF OF THEOREM 2:** The proof is by induction. Let $Q_{t,T}(z, x)$ denote the function $Q_t(z, x)$ for a $T$-period problem. Then, from (9) and Lemma 3,

$$\frac{\partial Q_{t+1}(z, A)}{\partial z} = M(p_{xx})(p_{xx} + h)[1 - G(z, A)] - (c + h) + \arg(z, A)K(z_{t+1}, z)$$

$$> M(p_{xx})(p_{xx} + h)[1 - G(z, A)] - (c + h) = \frac{\partial Q_{t+1}(z, A)}{\partial z}.$$
Thus, if \( z_{1,2} \), which is the value of \( z \) that maximizes \( Q_{1,2}(z, A) \), is greater than \( z_{1,1} \), which is the value of \( z \) that maximizes \( Q_{1,1}(z, A) \), we can conclude, for the induction hypothesis, that for the same initial condition, \( z_{1,T} > z_{1,T-1} \). Then, from (9),

\[
\frac{\partial Q_{1,T+1}(z, A)}{\partial z} = M(p_{z,A})(\gamma + h)[1 - G(z, A)] - (c + h) + \alpha g(z, A)K(z_{2,T+1}, z)
\]

\[
> M(p_{z,A})(\gamma + h)[1 - G(z, A)] - (c + h) + \alpha g(z, A)K(z_{2, T}, z)
\]

\[
= \frac{\partial Q_{1,T}(z, A)}{\partial z}.
\]

The inequality follows by this reasoning: \( z_{2, T+1} \), which represents the conditionally optimal decision in the second period of a \( T + 1 \)-period problem given that \( z \) is the first-period decision and demand is not observed in the first period, can be interpreted instead as the optimal decision in the first period of a \( T \)-period problem in which \( z \) is the initial lower bound for \( x \). As such, \( z_{2, T+1} > z_{2, T} \) by the induction hypothesis. Thus, \( K(z_{2,T+1}, z) > K(z_{2, T}, z) \) by Lemma 3. Since \( \partial Q_{1,T+1}(z, A)/\partial z > \partial Q_{1,T}(z, A)/\partial z \), we conclude that \( z_{1,T+1} > z_{1, T} \).

**PROOF OF THEOREM 3:** Consider the behavior of \( \partial Q_{1}(z, x)/\partial z \) as a function of \( x \). From (9) and Table 4,

\[
\frac{\partial}{\partial x} \left( \frac{\partial Q_{1}(z, x)}{\partial z} \right) = M(p_{x,z})[1 - G(z, x)] \left( \frac{\partial p_{x,z}}{\partial x} + (p_{x,z} + h)\gamma T(x) \right)
\]

\[
+ M'(p_{x,z})(p_{x,z} + h)[1 - G(z, x)] - (c + h) \frac{\partial p_{x,z}}{\partial x} + \alpha g(z, x)K(z_{1,1}, z)\gamma T(x).
\]

In the additive demand case, \( M(p_{x,z}) = 1 \), \( M'(p_{x,z}) = 0 \), and \( \partial p_{x,z}/\partial x > 0 \). Therefore, the function \( \partial Q_{1}(z, x)/\partial z \) is strictly increasing in \( x \), which implies that \( \partial Q_{1}(z, x)/\partial z \) crosses the \( x \)-axis at most one time in the additive demand case.

In the multiplicative demand case, \( M(p_{x,z}) = y(p_{x,z}) \). For this case, it can be shown that

\[
\left( \frac{\partial}{\partial x} \left( \frac{\partial Q_{1}(z, x)}{\partial z} \right) \right)_{\partial Q_{1}(z, x)/\partial x = 0} = -y'(p_{x,z})(c + h)g(x)\Gamma(x),
\]

where

\[
\Gamma(x) = (p_{x,z} + h) - \frac{(c + h)z}{\mu(x) - \Theta(x, x)} \left( 1 + \gamma T(x)U(x) \right),
\]

\[
0 < T(x) \equiv 1 - \frac{\mu(x)}{\mu(x) - \Theta(x, x)} \left( 1 + \gamma T(x)U(x) \right) < 1,
\]

\[
0 < U(x) \equiv 1 - \frac{z(1 - G(z, x))}{\mu(x) - \Theta(x, x)} \left( 1 + \gamma T(x)U(x) \right) < 1,
\]

and \( y'(p_{x,z}) = \left( \partial y(p)/\partial p \right)_{p = p_{x,z}} \). In (a) of the proof of Lemma 3 it was shown that \( (p_{x,z} + h) > (c + h)z \). In addition, from Lemma 2, \( (p_{x,z} + h) > (c + h)z/[\mu(x) - \Theta(x, x)] \). Therefore,

\[
(a) \quad \Gamma(x) > \frac{(c + h)z}{\mu(x) - \Theta(x, x)} \left( 1 - \frac{1}{1 + \gamma T(x)U(x)} \right)
\]

\[
\gamma \geq \frac{1}{1 - \gamma T(x)U(x)},
\]

then \( \gamma \geq \gamma T(x)U(x) \), which implies that \( \Gamma(x) > 0 \), from (b). If \( \gamma < 1/(1 - \gamma T(x)U(x)) \), then \( \gamma T(x)U(x) < 1 \), which implies that \( \Gamma(x) > 0 \), from (a). Therefore, \( \Gamma(x) > 0 \). Consequently, \( \partial \partial Q_{1}(z, x)/\partial z/\partial z \partial g_{Q_{1}(z, x)}/\partial z = 0 > 0 \), which implies that \( \partial Q_{1}(z, x)/\partial z \) crosses the \( x \)-axis at most one time in the multiplicative demand case.

**PROOF OF THEOREM 4:** Defining \( Q_{1}^T(z, x) \) as the maximand given in (13) after substituting \( p = p_{x,z} \) and \( \Pi(z, p, x) = \Pi(z, x) \), and applying the relationships from Table 4, we can write expressions analogous to (9) and (10):

\[
\frac{\partial Q_{1}^T(z, x)}{\partial z} = (p_{x,z} + h)[1 - G(z, x)] - (c + h) + \alpha \int_{x}^{z} \frac{\partial V_{i+1}(u, \rho(z - u))}{\partial z} g(u, x) \, du + \alpha g(z, x)K^T(z_{1,1}, z)
\]

and

\[
K^T(z_{1,1}, z) \equiv V_{i+1}(z, \rho(0)) - ((p_{z_{i+1}, z} - c)D(p_{z_{i+1}, z}, z) + \alpha V_{i+2}(z, \rho(z_{i+1} - z))) + (c + h)(z_{i+1} - z).
\]

Given the logic presented in the proofs of Theorems 1–3, to prove Theorem 4, it suffices to show:

\[
(a) \quad \alpha \int_{x}^{z} \frac{\partial V_{i+1}(u, \rho(z - u))}{\partial z} g(u, x) \, du \geq 0,
\]
(b) $K^I(z, x) \geq 0$ as $z \to x$.
(c) $K^I(z, x) > 0$ and $\partial K^I(z, x)/\partial z > 0$ for $z > x$.
(d) the function $\partial Q^I_I(z, x)/\partial z$ crosses the $x$-axis at most one time for $z > x$.

Property (a): Given that $\partial V_I(u, I)/\partial I \geq 0$ and $d\rho(X)/dX \geq 0$, the result follows directly.

Property (b): Given that $\partial V_I(u, I)/\partial I \leq c$ and $\rho(X) \leq X$, we have $V_{I+2}(x, \rho(z-x)) \leq V_{I+2}(x, \rho(0)) + c(z-x)$. Also, $V_{I+2}(x, \rho(0))$ is equivalent to $V_{I+1}(x)$ from the perishable-product case. As such, from (6), it can be written as: $V_{I+1}(z, \rho(0)) = (p(x) - c)D(p(x), x) + \alpha V_{I+2}(x, \rho(0))$. Consequently, $K^I(z, x) \geq (p(x) - c)D(p(x), x) - (p_{zx} - c)D(p_{zx}, x) + (c + \alpha)(z - x)$. Finally, from Lemma 1, $p_{zx} = p^0(c) + x/2b = p(x)$. Therefore, as $z \to x$, $K^I(z, x) \geq 0$.

Property (c): From Table 4, if $z > x$, then $\mu(x) - \Theta(z, x) > x$, which implies that $p_{zx} > p(x)$. Therefore, $K^I(z, x) > 0$ follows analogously to the proof of property (b). Next, recall that $D(p, x) = y(p) + x$. Thus, applying Lemma 1,
\[ \frac{\partial K^I(z, x)}{\partial z} = [\mu(x) - \Theta(z, x) - x] \frac{\partial p_{zx}}{\partial x} + (c + h) - \alpha \frac{\partial V_{I+2}(x, \rho(z-x))}{\partial z} > (c + h) - \alpha \frac{\partial V_{I+2}(x, \rho(z-x))}{\partial z}. \]

But $\partial V_{I+2}(x, \rho(z-x))/\partial z \leq c(d\rho(z-x)/dz) \leq c$. Therefore, $K^I(z, x)/\partial z > 0$.

Property (d):
\[ \frac{\partial}{\partial z} \left[ \frac{\partial Q^I_I(z, x)}{\partial z} \right] = [1 - G(z, x)] \frac{\partial p_{zx}}{\partial x} - \alpha \frac{\partial V_{I+2}(x, \rho(z-x))}{\partial z} r(x) + \left[ \frac{\partial Q^I_I(z, x)}{\partial z} + (c + h) \right] r(x). \]

But $[1 - G(z, x)](\partial p_{zx}/\partial x) > 0$ and $\partial V_{I+1}(x, \rho(z-x))/\partial z \leq c$. Therefore, $\partial(\partial Q^I_I(z, x)/\partial z)/\partial x > 0$ when evaluated at $\partial Q^I_I(z, x)/\partial z = 0$, which implies that the function $\partial Q^I_I(z, x)/\partial z$ crosses the $x$-axis at most one time.

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