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STABILIZATION OF THE EXTENDED NONHOLONOMIC DOUBLE INTEGRATOR VIA LOGIC-BASED HYBRID CONTROL

António Pedro Aguiar [1] **António Pascoal**

Institute for Systems and Robotics, Instituto Superior Técnico, Torre Norte 8, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
Phone: +351-21-8418090, Fax: +351-21-8418291
E-mail: {antonio.aguiar,antonio}@isr.ist.utl.pt

Abstract: This paper derives a hybrid control law for an extended nonholonomic double integrator (ENDI) that captures the dynamics of a wheeled robot subject to force and torque inputs. A simple logic-based hybrid controller is proposed which yields global stability and convergence of the closed-loop system to an arbitrarily small neighborhood of the origin. This is achieved by mapping the state-space into a two dimensional closed positive quadrant space and dividing it into three overlapping regions where, for each region, a feedback law is conveniently designed. Convergence and stability of the closed-loop hybrid system are analyzed theoretically. An application is made to the control of a wheeled mobile robot of the unicycle-type. Simulation results are presented that illustrate the performance of the hybrid control law.

Keywords: Nonholonomic Systems, Hybrid Control, Point Stabilization, Autonomous Mobile Robots

1. INTRODUCTION

The control of nonholonomic systems has been the subject of considerable research effort over the last few years. The reason for this trend is threefold: i) there are a large number of mechanical systems that have non integrable constraints such as robot manipulators, mobile robots, wheeled vehicles, and space and underwater robots; ii) there is considerable challenge in the synthesis of control laws for systems that are not transformable into linear control problems in any meaningful way and, iii) as pointed out in a famous paper of Brockett (Brockett, 1983), nonholonomic systems cannot be stabilized by continuously differentiable, time invariant, state feedback control laws. To overcome the limitations imposed by the celebrated Brockett's result, a number of approaches have been proposed for stabilization of nonholonomic control systems to equilibrium points. See (Kolmanovsky and McClamroch, 1995) and the references therein for a comprehensive survey of the field. Among the proposed solutions are smooth time-varying controllers (Godhavn and Egeland, 1997; Samson, 1995), discontinuous or piecewise smooth control laws (Canudas de Wit and Sørdalen, 1992; Bloch and Drakunov, 1994; Astolfi, 1999), and hybrid controllers (Hespanha, 1996). Specially attractive are discontinuous control laws, which in some cases can overcome the complexity and lack of good performance (e.g., low rates of convergence and oscillating trajectories) that are often associated with time-varying control strategies. The reader is referred to (Astolfi, 1999) for a discussion of this interesting circle of ideas.

Despite the vast amount of papers published on the stabilization of nonholonomic systems, the majority has concentrated on kinematic models of mechanical systems controlled directly by velocity inputs, while less attention has been paid to the control of nonholonomic dynamical mechanical systems where forces and torques are the actual inputs. See for example (M'Closkey and Murray, 1994) where the authors extended time-varying exponential stabiliz-

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ers to dynamic nonholonomic systems.

The main contribution of this paper is the derivation of a hybrid control law for an extended nonholonomic double integrator (ENDI) that captures the dynamics of a wheeled robot subject to force and torque inputs. The methodology proposed for control system design was inspired by the work of Hespanha (Hespanha, 1996) where a hybrid controller for the nonholonomic integrator (Brockett, 1983) was derived. The new control law solves the problem of global convergence and stabilization of the ENDI system to an arbitrarily small neighborhood of the origin. This is achieved by mapping the state-space into a two dimensional closed positive quadrant space and dividing it into three overlapping regions where, for each region, a feedback law is conveniently designed. Convergence and stability of the closed-loop hybrid system are analyzed theoretically. An application is made to the control of a wheeled mobile robot of the unicycle-type. Simulation results are presented that illustrate the performance of the hybrid control law.

The paper is organized as follows: section 2 introduces the extended nonholonomic double integrator. In section 3 a simple hybrid controller is proposed, and in section 4 closed-loop stability and convergence to the origin are analyzed. Section 5 illustrates the application of the hybrid control law to point stabilization of a wheeled mobile robot. Finally, discussions and recommendations for further research are given in section 6.

2. THE EXTENDED NONHOLONOMIC DOUBLE INTEGRATOR

In (Brockett, 1983) Brockett introduced the so called *nonholonomic integrator* system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{1}$$

where $[x_1, x_2, x_3]' \in \mathbb{R}^3$ is the state vector and $[u_1, u_2]' \in \mathbb{R}^2$ is a two-dimensional input. It can be shown that any kinematic completely nonholonomic system (e.g., the kinematic model of a wheeled mobile robot of the unicycle type) with three states and two inputs can be converted into the above form by a local coordinate transformation. The nonholonomic integrator displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design. See for example (Bloch and Drakunov, 1994) for the description of sliding mode feedback control laws for stabilization of the nonholonomic integrator and (Astolfi, 1998) for the derivation of a family of discontinuous controllers that almost exponentially stabilizes (1) in an open and dense set. See also (Hespanha, 1996) where a new methodology for nonholonomic integrator stabilization was proposed by resorting to a hybrid, logic based switching control law.

The nonholonomic integrator model fails to capture the case where both the kinematics and dynamics of a wheeled robot must be taken into account. To tackle this realistic case, the nonholonomic integrator model must be extended. This is done in section 5 where it is shown that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 \dot{x}_2 - x_2 \dot{x}_1 \end{aligned} \tag{2}$$

where $x \triangleq [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2]' \in \mathbb{R}^5$ is the state vector and $u \triangleq [u_1, u_2]' \in \mathbb{R}^2$ is a two-dimensional control vector. In this paper, system (2) will be referred to as *the extended nonholonomic double integrator* (ENDI) and will be used as a prototype system in the development that follows.

3. HYBRID CONTROLLER DESIGN

This section proposes a simple piecewise smooth controller to stabilize the ENDI system that borrows from hybrid system theory. Hybrid systems are specially suited to deal with the combination of continuous dynamics and discrete events. The literature on hybrid systems is extensive and discusses different modeling techniques. In (Ye *et al.*, 1998) Ye, Michel, and Hou formulated a model for hybrid dynamical systems that covers a very large class of systems and is suitable for qualitative analysis. They have also defined several types of Lyapunov-like stability concepts for an invariant set and established sufficient and necessary conditions (converse theorems) for these types of stability. Branicky, Borkar, and Mitter (Branicky *et al.*, 1994) proposed a very general framework for hybrid control problems that encompasses several types of such hybrid phenomena. See also (Branicky, 1998) where several tools for the analysis and synthesis of hybrid systems were developed.

In this paper, a continuous-time hybrid system Σ is defined as follows:

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t), t\big), \qquad t \geq t_0 \tag{3a}$$

$$\sigma(t) = \phi\left(x(t), \sigma(t^-)\right) \tag{3b}$$

where $\sigma(t) \in \mathcal{I} \triangleq \{1, \ldots, N\}$ and $x(t) \in \mathcal{X} \triangleq \cup_{\sigma=1}^{N} \mathcal{X}_\sigma \subset \mathbb{R}^n$. Here, the differential equation (3a) models the continuous dynamics, where the vector fields $f_\sigma : \mathcal{X}_\sigma \times \mathbb{R}^+ \to \mathcal{X}$, $\sigma \in \mathcal{I}$ are each locally Lipschitz continuous maps from $\mathcal{X}_\sigma$ to $\mathcal{X}$. The algebraic equation (3b), where $\phi : \mathcal{X} \times \mathcal{I} \to \mathcal{I}$, models the state of the decision-making logic. The discrete state $\sigma(t)$ is piecewise constant. The notation t^- indicates that the discrete state is piecewise continuous from the right. The dynamics of the system Σ can now be described as follows: starting at (x_0, i) with $x_0 \in \mathcal{R}_i \subset \mathcal{X}_i$, the continuous state trajectory $x(t)$ evolves according to $\dot{x} = f_i(x, t)$. When $\phi\big(x(\cdot), i\big)$

becomes equal to $j \neq i$, (and this could only happen when $x(\cdot)$ hits the set $\mathcal{X}\backslash\mathcal{R}_i$), the continuous dynamics switches to $\dot{x} = f_j(x,t)$, from which the process continues. As in (Hespanha, 1996), the "logical dynamics" will be determined, recursively by equation (3b) with $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$ where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ from below as $\tau \to t$ and the transition function ϕ is defined by

$$\phi(x,\sigma) = \begin{cases} \sigma & \text{if } x \in \mathcal{R}_\sigma, \\ \max_{\mathcal{I}}\{k : x \in \mathcal{R}_k\} & \text{otherwise.} \end{cases} \quad (4)$$

The signal $\sigma(t)$ can be also generated according to the diagram in Figure 1.

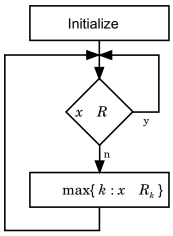

Fig. 1. Switching Logic.

Consider now the ENDI system (2). When the state variables x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and, consequently, x_3 will remain constant. Thus a possible strategy to steer an initial state to the vicinity of the origin is the following (see (Hespanha, 1996) where similar ideas were applied to the control of the nonholonomic integrator): i) first, make the state variable x_3 converge to zero while keeping x_1 and x_2 away from the axis $x_1 = x_2 = 0$; ii) next, freeze x_3 ($\dot{x}_3 = 0$), and force x_1 and x_2 to converge to zero.

In order to derive a hybrid controller for the ENDI, it is convenient to define the function $W(\cdot) : \mathbb{R}^4 \to \Omega \subset \mathbb{R}^2$ that maps the state-space coordinates $(x_1, x_2, x_3, \dot{x}_3)' \in \mathbb{R}^4$ into the two-dimensional closed positive quadrant [2] space Ω

$$\omega \triangleq [\omega_1, \omega_2]' = W(x) = \left[s^2, (x_1)^2 + (x_2)^2\right]',$$

where $s = \dot{x}_3 + \lambda x_3$ and λ is a strictly positive constant. This mapping has several properties, which are listed in the following lemma.

Lemma 1. The mapping $W(\cdot) : \mathbb{R}^4 \to \Omega \subset \mathbb{R}^2$ has the following properties:

(1) $W(0) = 0$.
(2) if w converges to zero as $t \to \infty$, then x also converges to zero as $t \to \infty$.
(3) if $x_3(t_0) = 0$ and $\omega_1 \leq \epsilon$ for all $t \geq t_0$, then $|x_3(t)| \leq \frac{\sqrt{\epsilon}}{\lambda}$ for all $t \geq t_0$. For the case where $x_3(t_0) \neq 0$, the bound of $x_3(t)$ is given by

$$|x_3(t)| \leq e^{-\lambda(t-t_0)}|x_3(t_0)| + \frac{\sqrt{\epsilon}}{\lambda}.$$

[2] The closed positive quadrant of $\mathbb{R}^2$ is the set $\Omega = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 \geq 0, \omega_2 \geq 0\}$.

Divide now Ω into three overlapping regions (see Figure 2)

$$\begin{aligned} \mathcal{R}_1 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \leq \gamma_2\} \\ \mathcal{R}_2 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \geq \gamma_1\} \\ \mathcal{R}_3 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 \leq \epsilon_2\} \end{aligned} \quad (5)$$

where $\epsilon_2 > \epsilon_1 > 0$ and $\gamma_2 > \gamma_1 > 0$.

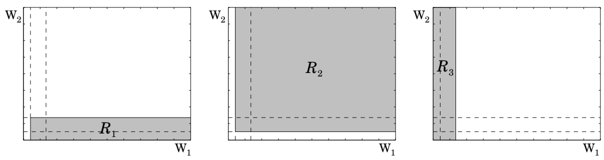

Fig. 2. Definition of the regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$.

Motivated by the work in (Hespanha, 1996), consider the following dynamical system as a candidate control law to steer the ENDI trajectories to a small neighborhood of the origin:

$$u = g_\sigma(x), \quad (6)$$

where the vector fields $g_\sigma : \mathbb{R}^5 \to \mathbb{R}^2$, $\sigma \in \mathcal{I} = \{1,2,3\}$ are given by

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 \\ -\lambda\dot{x}_2 + x_2 \end{bmatrix}, g_2(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + x_2 s \\ -\lambda\dot{x}_2 + x_2 - x_1 s \end{bmatrix},$$

$$g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 - x_1 \\ -\lambda\dot{x}_2 - x_2 \end{bmatrix}. \quad (7)$$

σ is a piecewise constant switching signal taking values in $\mathcal{I} = \{1,2,3\}$, and is determined recursively by

$$\sigma(t) = \phi\left(\omega(t), \sigma^-(t)\right), \quad \sigma^-(t_0) = \sigma_0 \in \mathcal{I} \quad (8)$$

where the transition function is defined according to (4). The control laws for each region were designed according to the following simple rule: while $\sigma = 1$, $\omega_1(t)$ must decrease or remain constant and $\omega_2(t)$ must grow without bound as $t \to \infty$; when $\sigma = 2$, $\omega_1(t)$ must decrease and reach a determined bound in finite time; and finally when $\sigma = 3$, $\omega_1(t)$ must again remain constant and $\omega_2(t)$ must converge to zero. A sketch of a typical trajectory of W is shown in Figure 3. The region that is the intersection of $\mathcal{R}_2$ and $\mathcal{R}_3$ can be seen as a hysteresis region. Its aim is to avoid the possibility of infinitely fast chattering when ω_1 is near ϵ_1.

Remark 2. For $\sigma = 1$, if λ does not satisfy the relation $-\frac{1}{2}\left(1 + \sqrt{\lambda^2+4}\right)x_1(t_0) \neq \dot{x}_1(t_0) \vee -\frac{1}{2}\left(1 + \sqrt{\lambda^2+4}\right)x_2(t_0) \neq \dot{x}_2(t_0)$, then the unstable mode of the corresponded closed-loop system is not excited. In that case, $g_1(x)$ has to be modified to $g_1(x) = \left[\begin{smallmatrix} -\lambda\dot{x}_1 + x_1 + \mathrm{sgn}(c_2)\,\mathrm{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - \mathrm{sgn}(c_1)\,\mathrm{sgn}(s) \end{smallmatrix}\right]$ where $\mathrm{sgn}(x) = 1$ if $x \geq 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$, $c_i = \frac{\dot{x}_i(t_0) - x_i(t_0)s_1}{s_2 - s_1}$, $i = 1, 2$, and $s_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\lambda^2+4}$.

4. STABILITY ANALYSIS

In this section stability and performance analysis of the closed-loop system consisting of the ENDI

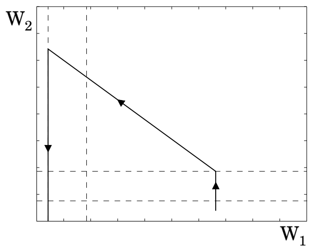

Fig. 3. Sketch of the image by $W : \mathbb{R}^4 \to \Omega$ of a typical trajectory.

system and the control law proposed are analyzed. To do this, it is necessary to extend the usual definition of Lyapunov stability for hybrid systems.

Definition 3. The equilibrium point $x = 0$ of the hybrid system Σ is Lyapunov stable if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \delta$, the solution $\{x(t), \sigma(t)\}$ satisfies $\|x(t)\| < \epsilon$, for all $t \geq t_0$. If in the above definition δ is independent of t_0, i.e., $\delta = \delta(\epsilon)$, then the origin is said to be uniformly stable.

The following theorem establishes the main result of this section.

Theorem 4. Consider the hybrid system Σ described by (2), (6)-(8), and (4). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ. Then,

1. $\{x(t), \sigma(t)\}$ is the unique solution that is defined for all $t \geq t_0$;
2. for any set of initial conditions $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$, there exists a finite time $T \geq t_0$ such that for $t > T$ the state variables $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$, and $\dot{x}_2(t)$ converge uniformly exponentially to zero, and $\omega_1(t) \leq \epsilon_2$, where $\epsilon_2 > 0$ is a controller parameter that can be chosen arbitrarily small;
3. the origin $x(t) = 0$ is a Lyapunov uniformly stable equilibrium point of Σ.

Proof. In the sequel the following notation is required: given a set $\mathcal{R} \subset \mathbb{R}^n$, its closure and boundary are denoted by $\overline{R}$ and $\partial\mathcal{R}$ respectively. $\mathcal{B}_\delta(x)$ denotes an open ball of radius $\delta > 0$ centered at x.

Uniqueness - Proof omitted.

Convergence

A proof of convergence can be given, based on the five claims below.

Claim 1. There exists a finite time $t_{\sigma_1} \geq t_0$ such that for all $t \geq t_{\sigma_1}$, $\omega(t) \notin \mathcal{R}_1 \backslash \mathcal{R}_2$.

Proof. Consider first that $\omega(t_0) \in \mathcal{R}_1 \backslash \mathcal{R}_2$ and suppose by contradiction that $\omega(t)$ remains in $\mathcal{R}_1 \backslash \mathcal{R}_2$ for all $t \geq t_0$. Since $\omega(t_0) \in \mathcal{R}_1 \backslash \mathcal{R}_2$, then $\sigma(t_0) = 1$, and $\sigma(t)$ will always be equal to 1 (since $\omega(t)$ never leaves $\mathcal{R}_1 \backslash \mathcal{R}_2$ by the contradiction hypothesis). Therefore, the closed-loop equation is $\dot{x} = g_1(x)$, and it can be checked that for this case $\dot{\omega}_1(t) \leq 0$ for all $t \geq t_0$ and $\omega_2(t) \to \infty$ as $t \to \infty$. Consequently, ω will leave the region $\mathcal{R}_1 \backslash \mathcal{R}_2$ which is a contradiction. The remainder of the proof shows that $\omega(t)$ will remain outside $\mathcal{R}_1 \backslash \mathcal{R}_2$ for $t \geq t_{\sigma_1}$. Let $t_{\sigma_1} \triangleq \sup\{t \in [t_0, \infty) : \sigma(t) = 1 \wedge \omega(t) \in \partial\mathcal{R}_1 \backslash \mathcal{R}_2\}$. When $\omega(t_{\sigma_1}) \in \partial\mathcal{R}_1 \backslash \mathcal{R}_2$, $\sigma(t_{\sigma_1}) = 1$ and σ will remain constant until the next switch, which must occur after some positive time interval, say $\delta > 0$. Therefore, for $t \in [t_{\sigma_1}, t_{\sigma_1} + \delta]$ one has $\dot{\omega}_1 \leq 0$ and $\dot{\omega}_2 > 0$, which shows that the velocity vector points (non strictly) to the outside of $\mathcal{R}_1 \backslash \mathcal{R}_2$. Thus, one can conclude that $\omega(t)$ will remain outside $\mathcal{R}_1 \backslash \mathcal{R}_2$ for $t \geq t_{\sigma_1}$.

Claim 2. For any $t_{\sigma_1} \geq t_0$ such that $\sigma(t_{\sigma_1}) = 1$, there exists a finite time $t_{\sigma_2} \geq t_{\sigma_1}$ such that $\sigma(t_{\sigma_2}) = 2$.

Proof. For $\sigma = 1$, $\dot{\omega}_1 \leq 0$ and $\omega_2(t) \to \infty$ as $t \to \infty$. Thus, there exists a finite time t_{σ_2} such $\omega_2(t_{\sigma_2}) = \gamma_2$, which implies that for $t = t_{\sigma_2}$, σ switches to 2.

Claim 3. For any t_{σ_2} such that $\sigma(t_{\sigma_2}) = 2$, there exists a positive time interval $\tau > 0$ such that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$

$$\omega_1(t_{\sigma_2} + \tau) \leq \omega_1(t_{\sigma_2}) e^{-2\gamma_1 \tau}.$$

Proof. For $\sigma(t_{\sigma_2}) = 2$, the closed-loop equations of the hybrid system Σ are given by $\dot{x} = g_2(x)$. Since $g_2(x)$ is continuous in x, the solution $x(t)$ will be continuously differentiable and, consequently, there exists $\tau > 0$ such that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$, $\omega(t)$ will be inside $\mathcal{R}_2$ and $\sigma(t) = 2$. This means that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$,

$$\begin{aligned} \dot{\omega}_1 &= -2\omega_2\omega_1 \\ \omega_2 &\geq \gamma_1. \end{aligned}$$

Hence, $\omega_1(t_{\sigma_2} + \tau) \leq \omega_1(t_{\sigma_2}) e^{-2\gamma_1 \tau}$.

Claim 4. There exists a finite time $t_{\sigma_3} \geq t_0$ such that for all $t \geq t_{\sigma_3}$, $\omega(t) \in \mathcal{R}_3$.

Proof. This claim will be proven in two steps. First, it will be shown that for any $\omega(t_0) \in \Omega \backslash \mathcal{R}_3$, there exists a time $t_{\sigma_3} \geq t_0$ such that $\omega(t_{\sigma_3}) \in \mathcal{R}_3$ and second, that if $\omega(t_{\sigma_3}) \in \mathcal{R}_3$ then $\omega(t) \in R_3$ for all $t \geq t_{\sigma_3}$. Consider first that $\omega(t_0) \in \Omega \backslash \mathcal{R}_3$. Then, the objective is to prove that ω_1 will reach in finite time the boundary $\omega_1 = \epsilon_2$, where σ can only takes the value 1 or 2. From claim 1, it follows that after a finite time t_{σ_1}, $\omega_2(t) > \gamma_1$ (while $\omega \in \Omega \backslash \mathcal{R}_3$). Since the dynamics of ω_1 are given by

$$\dot{\omega}_1 = \begin{cases} \leq 0 & \sigma = 1, \\ -2\omega_2\omega_1 & \sigma = 2. \end{cases}$$

from claim 2 and 3 it can be concluded that there exists a finite time t_{σ_3} for which $\omega(t_{\sigma_3}) \in \mathcal{R}_3$. To conclude the proof, it remains to show that $\omega(t)$ will be always inside $\mathcal{R}_3$ for $t \geq t_{\sigma_3}$. This is easily proved due to the fact that when $\omega(t_{\sigma_3}) \in \partial\mathcal{R}_3$, $\sigma \in \mathcal{I}$ will remain constant for at least some positive time interval (say $\delta > 0$). Therefore for $t \in [t_{\sigma_3}, t_{\sigma_3} + \delta]$

$$\dot{\omega}_1 = \begin{cases} \leq 0 & \sigma = 1, \\ -2\omega_2\omega_1 & \sigma = 2, \\ 0 & \sigma = 3. \end{cases}$$

which shows that the velocity vector does in fact point (non strictly) to the inside of $\bar{\mathcal{R}}_3$ and consequently $\mathcal{R}_3$ is a positively invariant set.

Claim 5. For $t \geq T$, $\omega_1(t) \leq \epsilon_2$ and $\omega_2(t)$ converges exponential to zero as $t \to \infty$.

Proof. From claim 4, $\omega(t) \in \mathcal{R}_3$ for all $t \geq t_{\sigma_3}$. Thus, $\omega_1(t) \leq \epsilon_2$. Moreover, there exists a finite time $T \geq t_0$ such that for all $t \geq T$ $\omega(t) \in \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$. The proof of this follows *mutatis mutandi* the one given for claim 4. Hence, since when $\omega(t) \in \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$, $\sigma(t) = 3$, the closed-loop equation are given by $\dot{x} = g_3(x)$ which shows that $\omega_2(t)$ converges exponential to zero as $t \to \infty$.

Stability - Proof omitted.

This concludes the outline of the proof of theorem 4. □

5. STABILIZATION OF A WHEELED MOBILE ROBOT OF UNICYCLE-TYPE

This section illustrates an application of the hybrid control law developed to a mobile robot. Consider the wheeled mobile robot of the unicycle type, shown in Figure 4. The vehicle is equipped with two identical, parallel, and nondeformable rear wheels which are controlled independently by motors, and a front free wheel. It is assumed that the plane of each wheel is perpendicular to the ground and that the contact between the wheels and the ground is pure rolling and nonslipping, i.e., the velocity of the center of mass of the robot v is orthogonal to the rear wheels axis [3]. It is also assumed that the masses and inertias of the wheels are negligible and that the center of mass of the mobile robot is located in the middle of the axis connecting the rear wheels. Each rear wheel is powered by a motor which generates a control torque τ_i, $i = 1, 2$. The goal is to park the wheeled mobile robot at a point with a desired posture.

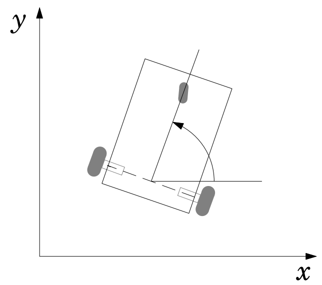

Fig. 4. A wheeled mobile robot of unicycle type.

5.1 *Model*

The kinematics and dynamics of the mobile robot are modeled by the equations

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \\ m\dot{v} &= F \\ I\dot{\omega} &= N \end{aligned}$$

where x and y denote the position of the wheel axis center, and θ is the robot orientation with respect to the x-axis. The symbols v and ω are the linear and angular mobile robot velocities, respectively. The control inputs are the pushing force F and the steering torque N, which are related to the wheel motor torques in the following manner:

$$\begin{aligned} F &= \frac{1}{R}(\tau_1 + \tau_2) \\ N &= \frac{L}{R}(\tau_1 - \tau_2) \end{aligned}$$

where R is the radius of the rear wheels and $2L$ is the length of the axis between the two rear wheels. The symbols m and I denote the mass and the moment of inertia of the mobile robot, respectively.

5.2 *Coordinate Transformation*

Consider the state and control transformation defined by

$$\begin{aligned} z_1 &= \theta, \\ z_2 &= x\cos\theta + y\sin\theta, \\ z_3 &= x\sin\theta - y\cos\theta, \\ u_1 &= \frac{N}{I}, \\ u_2 &= \frac{F}{m} - \frac{N}{I}x_3 - \omega^2 x_2. \end{aligned}$$

This transformation leads to a representation of the robot dynamics in the extended power form

$$\begin{aligned} \ddot{z}_1 &= u_1 \\ \ddot{z}_2 &= u_2 \\ \dot{z}_3 &= \dot{z}_1 z_2 \end{aligned} \tag{9}$$

It can be easily seen that applying the coordinate transformation

$$\begin{aligned} x_1 &= z_1, \\ x_2 &= z_2, \\ x_3 &= -2z_3 + z_1 z_2, \end{aligned}$$

to (9) yields the ENDI system in equation (2).

5.3 *Simulation results*

The objective was to park the vehicle at position $(x, y) = (0, 0)$ with heading $\theta = 0$. Several computer simulations were carried out with the controller designed in section 3, and applying the coordinate transformation of section 5.2. The control parameters were chosen to be $\lambda = 1.0$, $\epsilon_1 = 0.001$, $\epsilon_2 = 0.2$, $\gamma_1 = 1.0$, and $\gamma_2 = 2.0$. The regions were defined according to equations (5). The mass and the moment of inertia of the mobile robot are unitary. Figure 5 shows the mobile robot trajectory and figure 6

[3] By assuming that the wheels do not slide, a nonholonomic constraint on the motion of the mobile robot of the form $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$ is imposed.

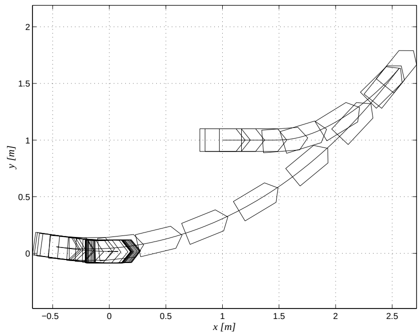

Fig. 5. Vehicle trajectory.

the time evolution of the states $x(t)$, $y(t)$, and $\theta(t)$ for the initial condition $x(0) = 1\,m$, $y(0) = 1\,m$, $\theta(0) = 0\,rad$, $v(0) = 0\,m/s$, and $\omega(0) = 0\,rad/s$. To better understand the performance of the hybrid control law, figure 7 displays the time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$. From the figure, one can see that while the state $\omega(t)$ is in the region $\mathcal{R}_1$, ω_2 grows in order to abandon that region and ω_1 remains constant. Then, σ switches to 2 and ω_1 starts to converge to zero, until it reaches the boundary $\omega_1 = \epsilon_1$. At that moment σ switches to 3, which implies that ω_2 converges to the origin.

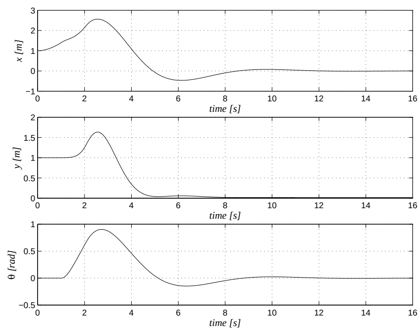

Fig. 6. Time evolution of the position variables $x(t)$ and $y(t)$, and the orientation variable $\theta(t)$.

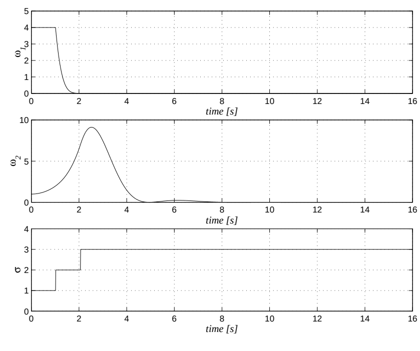

Fig. 7. Time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$.

6. CONCLUSIONS

A hybrid control law was derived for an extended nonholonomic double integrator (ENDI) that captures the dynamics of a wheeled robot subject to force and torque inputs. A simple logic-based hybrid controller was proposed that yields global stability and convergence of the closed-loop system to an arbitrarily small neighborhood of the origin. Convergence and stability of the closed-loop hybrid system were analyzed theoretically. An application was made to the control of a wheeled mobile robot of the unicycle-type. Simulation results show that the control objectives were achieved successfully. Future research issues will aim at generalizing the hybrid controller structure to a larger class of systems such as systems in chained form, as well as to systems subjected to input and rate limitations on the control signals. Another open problem that remains the subject of ongoing research efforts is the control and analysis of mechanical nonholonomic systems in the presence of noisy measurements and observer dynamics.

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