A distributed Model Predictive Control scheme for coordinated output regulation

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Abstract: This paper addresses the coordinated output regulation control problem. Consider a network of agents with associated output equations, where the latter is a function of the state of the agent and a coordination vector. Each agent can access its state, its coordination vector, and the coordination vectors of the neighboring agents. We wish to design a distributed control law that steers the output signals to the origin, while simultaneously driving the coordination vectors of the agents of the network to consensus. The proposed model predictive control scheme builds on a pre-existing auxiliary consensus control law to design a performance index that combines the output regulation objective with the consensus objective. A numerical simulation shows the effectiveness of the proposed scheme to solve the cooperative path following control problem for a network of under-actuated vehicles.

Keywords: Model predictive control; output regulation; multi-agent systems; distributed control; constrained systems; nonlinear control systems; cooperative path following; networked robotic system modeling and control; autonomous robotic systems; guidance navigation and control

1. INTRODUCTION

The coordinated output regulation problem stands at the intersection of the consensus problem in multi-agent systems (MASs) and the output stabilization problem.

The main objective of the consensus problem is the design of a control law for the agents of a MAS to drive the state of each agent to a common value, by only using measurements of the states of neighboring agents. The high relevance of such problem motivated an extensive literature. See, e.g., Bertsekas and Tsitsiklis (1989); Olfati-Saber and Murray (2004); Olfati-Saber et al. (2007) for an introduction to the topic.

In a standard stabilization problem using a Model Predictive Control (MPC) scheme, the controller select among all the feasible finite horizon future input trajectories the one that minimizes a given performance index, applies the first part of the optimal input trajectory to the system, and iterate the process. We refer to Mayne et al. (2000); Rawlings and Mayne (2009); Mayne (2014) for a survey on MPC and to Scattolini (2009) for a classification of different MPC architectures in MAS. In this paper, we focus on distributed non-iterative MPC schemes.

One of the main challenges in the use of MPC for MAS stems from the fact that an agent only optimizes over its future input trajectory without knowledge of the behavior of the neighboring systems. To address this problem, in Dunbar and Murray (2006); Wang and Ong (2010), all the agents exchange their optimal predictions, obtained at the previous step, with the neighboring agents and adopt suitably designed constraints, called consistency constraints, to guarantee the new optimal prediction to be close to the one obtained from the previously exchanged trajectory. Such fact is then used in the analysis to obtain closed-loop guarantees. In Wang and Ong (2010), convergence to the minimal disturbance invariant set is guaranteed for the case of systems with noise. An interesting approach for the consensus of single-/double-integrator system is presented in Ferrari-Trecate et al. (2009). Here, choosing a performance index that penalizes the distance from the current value of the state to the barycenter of the convex hull formed with the state of the neighboring agents, the authors show that the state of the system is driven in the interior of the latter set and therefore, building on Moreau (2005), convergence to consensus is guaranteed.

The output regulation problem regards to the design of a control law to drive the output of the system to a desired trajectory or geometric path. We refer to Faulwasser and Findeisen (2015); Alessandretti et al. (2013) and reference therein for MPC solutions to these problems.

The goal of this paper is to provide a scheme able to perform output regulation of the output signal of the agents in a MAS, where such output is a function of a coordination parameter that we wish to drive to consensus. In contrast to the methods mentioned above, we build on a pre-existing auxiliary consensus law that acts as a reference for the desired control input for the coordination parameter. The proposed solution only requires measurements of the current coordination vector of the neighboring system and no consistency constraints are enforced. Conditions under which the coordinated output regulation can be solved with convergence guarantees are provided.
The remainder of the paper is organized as follows. Section 2 provides the definition of the control problem. Section 3 describes the proposed solution, followed by Section 4 with a numerical simulation where the proposed scheme is applied to solve the coordinated path following problem. The proofs are omitted due to space constraints.

2. PROBLEM DEFINITION

Consider a set of $n_T \in \mathbb{Z}_{\geq 1}$ continuous-time dynamical systems where the generic $i$-th system, $i \in \mathcal{I} := \{1, 2, \ldots, n_T\}$ is described as

$$\dot{x}^i[t] = f^i[t, x^i(t), u^i(t)], \quad x^i(t_0) = x^i_0, \quad t \geq t_0$$

with $x^i(t) \in \mathbb{R}^{n[i]}$ and $u^i(t) \in \mathcal{U}(t) \subseteq \mathbb{R}^{m[i]}$ denoting the state and input vectors at time $t \geq t_0$. The scalar $t_0 \in \mathbb{R}$ and the vector $x^i_0 \in \mathbb{R}^{n[i]}$ represent the initial time and state of the system, respectively, and the input $u^i(t)$ is constrained within the input constraint set $\mathcal{U}[i] : \mathbb{R}_{\geq 0} \Rightarrow \mathbb{R}^{m[i]}$.

Furthermore, each system includes a special internal state denoted as \textit{coordination vector} $\gamma^i(t) \in \mathbb{R}^{n_\gamma}$ that evolves with time according to

$$\dot{\gamma}^i(t) = v_d + u^i_{\gamma}(t), \quad \gamma^i(t_0) = \gamma^i_0, \quad t \geq t_0 \quad (1b)$$

for a constant predefined vector $v_d \in \mathbb{R}^{n_\gamma}$ and an input signal $u^i_{\gamma}(t) \in \mathbb{R}^{n_\gamma}$.

The systems communicate among each others according to a communication graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where the vertex set $\mathcal{V}$ collects all the indexes of the systems, that is, $\mathcal{V} = \mathcal{I}$, and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is such that $(i, j) \in \mathcal{E}$ if and only if the system $i$ can access $\gamma_j^i(t)$. Therefore, system $i$ can access from their neighborhoods $j \in \mathcal{N}[i] := \{j : (i, j) \in \mathcal{E}\}$ at time $t$ the coordination vectors $\gamma_{\mathcal{N}[i]}(t) := \{\gamma^j(t) : j \in \mathcal{N}[i]\}$.

The output of each system is defined as

$$y^i(t) = h^i(t, x^i(t), \gamma^i(t)) \in \mathcal{Y}(t) \quad (1c)$$

that is constrained within the output constrains set denoted by $\mathcal{Y}[i] : \mathbb{R}_{\geq 0} \Rightarrow \mathbb{R}^{m_0}$.

Given the above setup, we can now formulate the following problem:

\textbf{Problem 1.} (Coordinated output regulation). Design a control law for the input signals $u^i \in \mathcal{PC}(t_0, \infty)$\footnote{The term $\mathcal{C}(a, b)$ and $\mathcal{PC}(a, b)$ denotes the space of continuous and piecewise continuous trajectories, respectively, defined over $[a, b)$ or $(a, +\infty)$ for the case where $b = +\infty$.} and $u^i_{\gamma} \in \mathcal{PC}(t_0, \infty)$ such that for every $i \in \mathcal{I}$ the state vectors $x^i(t)$ and $y^i(t)$ are bounded for all $t \geq t_0$, and as time approaches infinity the following holds:

1. The output vector $y^i(t) \in \mathbb{R}^{p[i]}$ converges to the origin;
2. The network disagreement function
$$\phi(t) := \sum_{(i, j) \in \mathcal{E}} (\gamma^i(t) - \gamma^j(t))^2, \quad (1d)$$

converges to the origin;
3. The vector $\dot{\gamma}^i(t)$ converges to a predefined value $v_d \in \mathbb{R}^{n_\gamma}$.

\textbf{2.1 Example: Cooperative path-following}

An application that fits the proposed framework is the cooperative path-following problem described in this section. Let $I$ be an inertial coordinate frame and $B[i]$ be a body coordinate frame attached to the generic vehicle $i$. The pair $(p[i](t), R[i](t)) \in SE(2)$\footnote{For a given $n \in \mathbb{N}$, $SE(n)$ denotes the Cartesian product of $\mathbb{R}^n$ with the group $SO(n)$ of $n \times n$ rotation matrices and $se(n)$ denotes the Cartesian product of $\mathbb{R}^n$ with the space $so(n)$ of $n \times n$ skew-symmetric matrices.} denote the configuration of the vehicle, position and orientation, where $R[i](t)$ is the rotation matrix from body to inertial coordinates. Now, let $(v^i(\cdot, t), \Omega(\omega^i(\cdot))) \in SE(2)$ be the twist that defines the velocity of the vehicle, linear and angular, where the matrix $\Omega(\omega^i(t))$ is the skew-symmetric matrix associated to the angular velocity $\omega^i(t) \in \mathbb{R}$, defined as

$$\Omega(\omega) := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$
3. MAIN RESULT

This section introduces a sampled-data MPC controller that solves Problem 1. The key idea behind the proposed scheme is to combine the performance index from output tracking MPC with another term that rewards the consensus among the coordination vectors of the systems in the network. To this end, we rely on existing results on consensus control laws for discrete time systems, and in particular we assume the following:

Assumption 1. (Auxiliary consensus control law). Consider a set of discrete-time systems that communicate according to the same communication graph \( G = (V,E) \) introduced in Section 2, and satisfies

\[
\xi^{(i)}(k+1) = \xi^{(i)}(k) + k_{\text{con}}(\xi^{(i)}(k), \xi^{(N\setminus i)}(k)) + \eta^{(i)}(k)
\]

with \( i \in I \), where \( \xi^{(i)}(k) \in \mathbb{R}^{n_i} \), \( \eta^{(i)}(k) \in \mathbb{R}^{n_i} \), and \( \xi^{(N\setminus i)}(k) \) denote the \( i \)-th coordination vector, an external vector, and the coordination vectors from the neighborhood \( N^{(i)} \), respectively, at step \( k \in \mathbb{Z}_{\geq k_0} \), with \( \xi^{(i)}(k_0) = \xi^{(i)}_0 \in \mathbb{R}^{n_i} \) denoting the initial condition of the system \( i \) at the initial time step \( k_0 \in \mathbb{Z} \). If

\[
\|\eta^{(i)}(k)\| \leq a_{i\eta} e^{-\lambda_{\eta}(k-k_0)}
\]

for some constants \( \lambda_{\eta} > 0 \) and \( a_{i\eta} \geq 0 \) then:

(1) As \( k \to \infty \), the disagreement function

\[
\phi(k) = \sum_{(i,j) \in E} (\xi^{(i)}(k) - \xi^{(j)}(k))^2
\]

converges asymptotically to zero;

(2) The consensus control law \( k_{\text{con}} : \mathbb{R}^{n_N} \to U_c \) has a bounded output, i.e., \( U_c \subset \mathbb{R}^{n_N} \) is bounded;

(3) The control law satisfies

\[
\|k_{\text{con}}(\xi^{(i)}(k), \xi^{(N\setminus i)}(k))\| \leq \beta(\sum_{i \in I} \|\xi^{(i)}_0\|, k - k_0)
\]

for an integrable class-KL function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \). \( \square \)

Proposition 2. For \( n_s = 1 \) (scalar state) and \( G \) being a balanced strongly connected graph, the control law

\[
k_{\text{con}}(\xi^{(i)}(t), \xi^{(N\setminus i)}(t)) = \bar{e} \sum_{j \in N^{(i)}} a_{ij}(\bar{e} - \xi^{(j)}(t)),
\]

with \( \bar{e} \in (0, 1/\Delta) \), \( \Delta = \max_i (\sum_{j \neq i} a_{ij}) \), and \( A := [a_{ij}] \) being the associated adjacency matrix with \( a_{ij} > 0 \) for all \( j \in N^{(i)} \) and \( a_{ii} = 0 \) otherwise, satisfies Assumption 1. \( \square \)

To solve Problem 1, we propose a sampled-data MPC scheme when the control input is computed at the time instants

\[
T := \{ t_0, t_1, \ldots \}, \quad \delta_k := t_{k+1} - t_k \geq \delta \theta > 0 \quad (8)
\]

for all \( k \geq 0 \) and for a constant scalar \( \delta \theta > 0 \), and applied open-loop to the system within the generic intervals \( [t_k, t_{k+1}) \). Towards the formulation of the open-loop MPC optimization problem, we first use the discrete-time control law from Assumption 1 to build a piecewise linear continuous-time control signal that solves the pure coordination task (i.e., points 2 and 3 of Problem 1) by only using the values of \( \gamma^{(i)}(t), \gamma^{(N\setminus i)}(t) \) evaluated at the time instants \( t \in T \).

To this end, for the generic time \( t \geq t_0 \) with \( t \in [t_k, t_{k+1}) \) and \( \gamma^{(i)}(t) \in \mathbb{R}^{n_i} \), the auxiliary coordination input trajectory \( u^{(i)}_{\gamma,\text{aux}}(t) \in C(t, +\infty) \) is defined as

\[
u^{[\gamma],\text{aux}}_i(\tau) = \begin{cases} 
\frac{1}{\tau} & \text{for } \tau \in [t, t+\delta_k] \\
0 & \text{for } \tau > t+\delta_k
\end{cases}
\]

for all \( t \in T \) and \( \delta \) defined in (8).

Let \( \lfloor t \rfloor \) be the maximum sampling instant \( t_k \in T \) smaller or equal to \( t \), i.e., \( \lfloor t \rfloor = \max_{k \in \mathbb{Z}_{>0}} \lfloor t_k \in T : t_k \leq t \rfloor \). It is worth noticing that, by Assumption 1, the control signal \( u^{(i)}_\gamma(t) = u^{(i)}_{\gamma,\text{aux}}(t) \) (t) points 2 and 3 of Problem 1. Because of this, such control law is used in the definition of the open-loop MPC problem that follows as an auxiliary controller for the coordination task.

Definition 3. (Open-loop MPC problem). Given the tuple of parameters \( p = (t, x^{(i)}_0, \gamma^{(i)}, \gamma^{(N\setminus i)}, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_N} \times \lambda^{(N\setminus i)} \times \mathbb{R}^{n_N} \), an horizon length \( T \in \mathbb{R}_{>0} \), and the auxiliary signal \( u^{(i)}_{\gamma,\text{aux}} \in C(t, +\infty) \) from (9), the open-loop MPC optimization problem \( P(p) \) consists in finding the optimal control signals \( \bar{u}^{(i)} \in \mathcal{PC}(t, T) \) and \( \bar{v}^{(i)} = PC(t, t+T) \) that solves

\[
J_T^{(i)}(p) = \min_{\bar{u}^{(i)} \in PC(t, t+T)} \bar{u}^{(i)}_\gamma \in \mathcal{P}(t, t+T)
\]

s.t.

\[
\dot{x}^{(i)}(\tau) = f^{(i)}(\tau, \bar{x}^{(i)}(\tau), \bar{u}^{(i)}(\tau)), \quad \bar{x}^{(i)}(t) = x^{(i)}_0, \\
\dot{\bar{y}}^{(i)}(\tau) = \bar{y}^{(i)}(\tau), \quad \bar{y}^{(i)}(t) = \gamma^{(i)}_0, \\
\dot{\bar{u}}^{(i)}(\tau) = \bar{u}^{(i)}_{\gamma,\text{aux}}(\tau) + \eta^{(i)}(\tau), \\
(10a)
\]

\[
\gamma^{(i)}(t + T), \bar{y}^{(i)}(t + T) \in \mathcal{Y}^{(i)}(t + T) \times \mathcal{U}^{(i)}(t) \times \gamma^{(i)}_0, \\
(10b)
\]

\[
\|\bar{y}^{(i)}(\tau)\| \leq a_{i\eta} e^{-\lambda_{\eta}(k-(t-\delta))}
\]

for all \( \tau \in [t, t+T) \) and with

\[
J_T^{(i)}(p, \bar{u}^{(i)}, \bar{v}^{(i)}) := \int_t^{t+T} \ell^{(i)}(\tau, \bar{x}^{(i)}(\tau), \bar{u}^{(i)}(\tau), \bar{y}^{(i)}(\tau)) d\tau + m^{(i)}(t + T, \bar{x}^{(i)}(t + T), \bar{y}^{(i)}(t + T))
\]

\[
+ \int_t^{t+T} \|\bar{y}^{(i)}(\tau)\| d\tau + \frac{1}{2} \bar{m}^{(i)}(\bar{y}^{(i)}(t + T))^2
\]

for four constants \( \lambda^{(i)}_\eta > 0 \), \( a_{i\eta} \geq 0 \), \( m^{(i)}_\eta \geq 0 \), and \( r^{(i)}_\eta \geq 0 \), and where the constraint (10c) can be omitted in the case of \( N^{(i)} = \emptyset \). The finite horizon cost \( J_T^{(i)}(\cdot) \), which corresponds to the performance index of the MPC controller, is composed of two stage costs, i.e., the regulation stage \( \ell^{(i)} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_N} \to \mathbb{R}_{\geq 0} \) and the consensus stage \( m^{(i)} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_N} \to \mathbb{R}_{\geq 0} \), and two terminal costs, i.e., the regulation terminal cost \( m^{(i)}_\eta : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0} \), and the consensus terminal cost \( \frac{1}{2} \bar{m}^{(i)} \bar{y}^{(i)}(t + T)^2 \). The regulation terminal cost is defined over the set of \( (t, x^{(i)}_0, \gamma^{(i)}) \) such that the

\[\text{the ball set of radius } r \geq 0 \text{ of suitable diameters.}\]
associated $y^{[i]}(t) \in Y^{[i]}_{\text{aux}}(t)$ belongs to the regulation terminal set $Y^{[i]}_{\text{aux}} : \mathbb{R}_{\geq t_0} \to \mathbb{R}^{n_i}$. Similarly, the consensus terminal cost is evaluated with $\eta^{[i]}(t + T) \text{ constrained in the consensus terminal set } B(\eta^{[i]}_{\text{f}})$. 

In the sequel, for the sake of clarity, we omit the explicit superscript label $[i]$ to the vehicle $i$ whenever it is clear from the context. We will also use the simplified notation of omitting the time dependence and in particular, for the dependent signals $\dot{x}$, $\ddot{u}$, $\dot{\gamma}$, $\ddot{v}$, a function $l(\cdot)$ evaluated as $l(\tau, x(\tau), \ddot{u}(\tau), \dot{\gamma}(\tau), \ddot{v}(\tau))$ is denoted by $l(\tau, x, \ddot{u}, \dot{\gamma}, \ddot{v})$ or $l(\tau)$. Moreover, for a generic time $t \geq t_0$, the superscript label $st$ is used to denote all the trajectories of a given signal associated with the optimal predictions of $P(t)$.

The proposed sampled-data MPC approach is obtained by solving the optimization problem in Definition 3 at every time sample $t_k \in T$ and applying the associated optimal input trajectories within the generic interval $[t_k, t_{k+1})$, with $k \in \mathbb{Z}_{\geq 0}$, as follows

$$u(t) = \hat{u}^*_{t_k}(t),$$

$$u_r(t) = \hat{u}_{r,\text{aux}1}(t) + \eta^*_{t_k}(t),$$

$$\eta(t) = \hat{\eta}^*_{t_k}(t),$$

where $\eta \in \mathbb{R}^{n_{\gamma}}$ denotes the state of the controller with initial condition $\eta_0 \in \mathbb{R}^{n_{\gamma}}$.

At this point, we are ready to state the sufficient conditions for the MPC controller (11) to solve Problem 1.

The first set of conditions is obtained by adapting the standard MPC sufficient conditions for state convergence to the origin in the state space, to the convergence of the output signal to the origin in the output space.

**Assumption 4.** The input constraint set $U(t)$ is uniformly bounded over time. Moreover, the function $f(\cdot)$ in (1a) is locally Lipschitz in $x$, piecewise continuous in $t$ and $u$, and bounded for bounded $x$ in the region of interest, i.e., the set $\{ \| f(t, x, u) \| : t \geq t_0, x \in \mathcal{X}, u \in U(t) \}$ is bounded for any bounded $\mathcal{X}$.  

**Assumption 5.** Consider the output defined in (1c).

1. The system (1) satisfies for all $t \geq t_0$ the input-output-to-state stability condition

$$\| x(t) \| \leq \beta_x(\| x_0 \|, t - t_0) + \sigma_u(\| u \|_{t_0, t}) + \sigma_x(\| y \|_{t_0, t}) + \sigma_x^2$$

for a class-$KL$ function $\beta_x : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, class-$K^4$ functions $\sigma_u, \sigma_y : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a constant scalar $\sigma_x \geq 0$;

2. The gradient of the right-hand side of the output equation (1c), i.e., $\nabla h(t, x, \gamma(t))$, is uniformly bounded over time for bounded values of $x$, i.e., for all $x$ with $\| x \| \leq B$, there exists a scalar $b_B > 0$, possibly dependent on $B > 0$, such that

$$\| \nabla h(t, x, \gamma) \| \leq b_B$$

for every time $t \geq t_0$ and $\gamma \in \mathbb{R}^{n_{\gamma}}$.  

It is worth noticing that when condition (12) holds with $\sigma_x = 0$ and if $\| u(t) \| \to 0$ as $\| y(t) \| \to 0$, a controller that solves Problem 1 would also drive the state of the system to the origin. However, since this is not always the desired behavior, as in the case of the considered example, the positive term $\sigma_x$ is introduced to guarantee only boundedness of the state trajectory. The point 2 of the latter assumption is rather general, and we refer to the illustrative example for more insight on when such condition holds.

**Assumption 6.** For any given signals $\gamma \in \mathcal{C}(t_0, +\infty)$ and $u_\ast \in \mathcal{PC}(t_0, +\infty)$ the following holds:

1. The output constraint set $Y(t)$ and the terminal set $0 \in Y_{\text{aux}}(t) \subseteq Y(t)$ are closed, connected, and contain the origin for all $t \geq t_0$. The input constraint set $U(t)$ is compact for all $t \geq t_0$ and uniformly bounded over time;

2. The constraint set $V\gamma(t)$ is compact, uniformly bounded over time and such that $B(\gamma, \eta_0) \subseteq V\gamma(t)$, for all $t \geq t_0$;

3. The regulation stage cost $l(\cdot)$ is zero with $\gamma(t) = 0$ and there exists a class-$K\infty$ function $\alpha_\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\| l(\tau, x, \ddot{u}, \dot{\gamma}, \ddot{v}) \| \geq \alpha_\gamma(\| y \|)$$

for all $(\tau, x, \ddot{u}, \dot{\gamma}, \ddot{v}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}}$;

4. The consensus cost function $l_\gamma(\cdot)$ is zero with $\gamma = 0$ and satisfies

$$l_\gamma(\eta, \nu, \gamma) \geq \alpha_\gamma(\| \eta \|) = m_\gamma \lambda_\gamma \eta^2$$

for all $(\eta, \nu, \gamma) \in \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}}$ and for two positive constants $m_\gamma$ and $\lambda_\gamma$;

5. For any given values of $(x, u, \gamma, u_\ast) \in \mathbb{R}^{n_{x}} \times \mathbb{R}^{n_{u}} \times \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}}$ the functions $l(t, x, u, \gamma, u_\ast)$ and $m(t, x, \gamma)$ are uniformly bounded over time $t \geq t_0$;

6. There exists a feasible auxiliary regulation control law $k_{\text{aux}} : \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n_{x}} \times \mathbb{R}^{n_{u}} \times \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}} \to \mathbb{R}^{n_{u}}$ such that, for the closed-loop system (1a) with $u(t) = k_{\text{aux}}(t, x, \gamma, u_\ast)$, with initial time and states $(\hat{t}, \hat{x}, \hat{\gamma}) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n_{x}} \times \mathbb{R}^{n_{\gamma}}$ and for all $u \in \mathcal{C}(t_0, +\infty)$ with $u_\ast(\tau) \in \{ \nu \} \oplus B(\nu)^5$, the input and output vector satisfy $u(t) \in U(t)$ and $y(t) \in Y_{\text{aux}}(t)$, the associated state $x(t)$ exists, is unique, and the condition

$$m(t + \delta, x(t + \delta), \gamma(t + \delta) - m(t, x(t), \gamma(t)) \leq - \int_{t}^{t + \delta} l(t)dt$$

holds for any $\delta > 0$.

**Assumption 7.** Consider the open-loop MPC optimization problem from Definition 3. The regulation stage cost $l(\cdot)$ and the regulation terminal cost $m(\cdot)$ are Lipschitz continuous on $(\gamma, u_\ast)$, i.e., there exists a pair of constants $C_l \geq 0$ and $C_m \geq 0$ such that

$$\| l(t, x, u, \gamma_1, u_\ast_1, \gamma_2, u_\ast_2) - l(t, x, u, \gamma_2, u_\ast_1, \gamma_2, u_\ast_2) \| \leq C_l \left[ \| \gamma_1 - \gamma_2 \| \right]$$

$$\| m(t, x, \gamma_1) - m(t, x, \gamma_2) \| \leq C_m \left[ \| \gamma_1 - \gamma_2 \| \right]$$

holds for any given $(t, x, u) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n_{x}} \times \mathbb{R}^{n_{\gamma}}, (\gamma_1, u_\ast_1, \gamma_2, u_\ast_2) \in \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}}$, and $(\gamma_1, u_\ast_2, \gamma_2, u_\ast_1) \in \mathbb{R}^{n_{\gamma}} \times \mathbb{R}^{n_{\gamma}}$.  

\[5\] The term $\oplus$ denotes the Minkowski sum, i.e., $A \oplus B = \{ a + b : a \in A, b \in B \}$ for two sets $A$ and $B$ of same dimentions.
In Assumption 7, the functions \( m(\cdot) \) and \( l(\cdot) \) are required to be Lipschitz only on the variables \((\gamma, u_3)\), which makes the assumption rather general. Moreover, the Lipschitz constants are not used in the design phase. Therefore, only their existence, and not their computation, is required.

The main result of this paper follows: Theorem 8. Consider a set of constrained systems (1) that communicates according to a communication network \( G = (V, E) \) as described in Section 2. If Assumptions 1-7 hold, then the proposed sampled-data MPC control law (11) solves Problem 1. The region of attraction of the proposed controller corresponds to the set of initial conditions of the system such that the open-loop MPC problem of Definition 3 is feasible. \( \square \)

4. NUMERICAL RESULTS

In this section, the above MPC results are used to solve the cooperative path following problem of Section 2.1.

Auxiliary regulation control law. Consider the output (4) with time derivative
\[
\dot{y} = -\Omega y - R^T \frac{\partial}{\partial \gamma} c_d(\gamma) \dot{\gamma} + \Delta u,
\]
and the Lyapunov-like function \( W = \|y\| \). Then, for any \( \epsilon_2 \neq 0 \), choosing \( u = k_{aux}(t, x, \gamma, u_3) \) with
\[
k_{aux}(t, x, \gamma, u_3) = \begin{cases} \Delta^{-1} \left( R^T \frac{\partial}{\partial \gamma} c_d(\gamma)(v_d + u_3) - K^T y \right), & \|y\| \neq 0 \\ \Delta^{-1} R^T \frac{\partial}{\partial \gamma} c_d(\gamma)(v_d + u_3), & \|y\| = 0 \end{cases}
\]
results in
\[
\dot{y} = \begin{cases} -\Omega y - K \frac{y}{\|y\|^2} - \lambda_{\min}(K), & \|y\| \neq 0 \\ 0, & \|y\| = 0 \end{cases}
\]
that implies
\[
W = \begin{cases} \frac{\|y\|^2}{\|y\|^2} - \lambda_{\min}(K), & \|y\| \neq 0 \\ 0, & \|y\| = 0 \end{cases}
\]
where we used the fact that \( \Omega \) is skew-symmetric and therefore \( \gamma y^T \Omega y = 0 \) for all \( y \in \mathbb{R}^2 \). As a consequence, the vector \( y \) converges in finite time to the origin as follows
\[
\|y(t)\| \leq \left\{ \begin{array}{ll} \|y(0)\| - \lambda_{\min}(K)(\tau - t), & \tau \in [t, t_0] \\ \|y(t_0)\|, & \tau > t_0 \end{array} \right. \quad (15)
\]
with \( t_0 := t + \frac{\|y(t)\|}{\lambda_{\min}(K)} \).

Regulation stage cost. Assumption 6 point 3 is satisfied by the regulation stage cost
\[
l(t, x, u, \gamma, u_3) = \|y\|^2 + \frac{||u - k_{aux}(t, x, \gamma, u_3)||_2^2}{2} \quad (16)
\]
for any \( Q > 0 \) and \( O \geq 0 \) of suitable dimensions and where we use the notation \( ||v||_2^2 := v^T A v \) for a vector \( v \) and a matrix \( A \) of suitable dimensions.

Regulation terminal cost. At this point let \((y_{aux}, u_{aux})\) be the pair of output and input trajectories of the system in closed-loop with the auxiliary regulation control law (14), starting at the time and output pair \((t, \hat{y})\). Then, combining (15) with (16), it follows that the associated regulation stage cost is upper bounded as
\[
l(\tau, y_{aux}, u_{aux}) \leq \hat{l}(\tau; \hat{y}) = \min \begin{cases} \lambda_{\max}(Q)||(\hat{y}) - \lambda_{\min}(K)(\tau - t)||^2, & \tau \leq \hat{t} + \frac{||\hat{y}||}{\lambda_{\min}(K)} \\ 0, & \tau > \hat{t} + \frac{||\hat{y}||}{\lambda_{\min}(K)} \end{cases}
\]
where \( \hat{l}(\cdot) \) satisfies
\[
\hat{l}(\tau; \hat{t} + \delta, y_{aux}(\hat{t} + \delta)) \leq \hat{l}(\tau; \hat{t}, \hat{y}) \leq \frac{\lambda_{\max}(Q)}{3\lambda_{\min}(K)} ||\hat{y}||^3 \quad (17)
\]
satisfies the terminal cost decrease (13) of Assumption 6 point 5.

Regulation terminal set. Next, we design a terminal set to satisfies Assumption 6 point 5. Specifically, we show that for a specific selection of the input constraint set, the auxiliary regulation control law is always feasible, and therefore we can omit the regulation terminal set. In fact, from (14) we have
\[
||y_{aux}(\tau)||_1 \leq \frac{||\Delta^{-1}||}{2} n_1 + \frac{||\Delta^{-1}K||}{2} n_1 := v_{max} \quad (18a)
\]
\[
||y_{aux}(\tau)||_2 \leq \frac{||\Delta^{-1}||}{2} n_1 + \frac{||\Delta^{-1}K||}{2} n_1 := \omega_{max} \quad (18b)
\]
where for a generic matrix \( A \), \([A]_i\) denotes its \( i \)-th row vector and
\[
n_1 := (||v_d||_1 + r_q) \sup_\gamma \left| \frac{\partial}{\partial \gamma} c_d(\gamma) \right| \left| \frac{\partial}{\partial \gamma^T} c_d(\gamma) \right| .
\]
Notice that we require the derivative of the desired path to be bounded. Thus, from (18), choosing and uniformly bounded \( U(t) \) with \( \{v, \omega : ||v|| \leq v_{max}, ||\omega|| \leq \omega_{max} \} \subseteq U(t) \) the auxiliary regulation controller is always feasible and therefore the regulation terminal set can be omitted, i.e., choosing \( y_{aux}(t) = \mathbb{R}^2 \) for all \( t \geq t_0 \). The chosen regulation terminal set, regulation terminal cost, and constraint sets satisfy Assumptions 6 points 1 and 5.

Consensus stage cost. The consensus stage cost as \( l_c(\eta, v_0) = 2Q_v \eta + v_0^T O_v \eta \) with \( Q_v > 0 \), \( \lambda_{\min}(Q_v) \geq m_\eta \lambda_\eta \), and \( O_v \geq 0 \) satisfies Assumption 6 point 4.

We can now invoke Theorem 8 to conclude that the proposed MPC control law solves the CPF problem.

Numerical results. We consider a setting with \( I = \{1, 2, 3\} \) where agent 2 communicates bidirectionally with agents 1 and 3, but agent 1 and 3 cannot communicate with each other. The desired trajectories are chosen as
\[
e_d^{[1]} = 5c_d^0(\gamma), \quad e_d^{[2]} = 10c_d^0(\gamma), \quad e_d^{[3]} = 15c_d^0(\gamma) \quad (19)
\]
controller drives the coordination vectors to minimize the output in the setting where $c_0^i(\gamma^i) \leq 0$ is as close as possible to $c_0^i(t)$ for each respective $i \in I$. In this phase, the disagreement function, originally equal to zero, increases to favour the output decrease. Then, once the output is decreased the controller drives the coordination vectors to consensus and the vehicle converge to the desired formation.

5. CONCLUSIONS

The paper proposed a distributed MPC scheme for coordinated output regulation. The proposed solution is non-iterative, in the sense that the exchange information among neighboring agents is performed only once every $t \in T$. The simulation results on cooperative path-following problem show how the proposed scheme optimally compromise between the consensus task and the output regulation.

REFERENCES


