A Class of Multi-parametric Quadratic Program with an Uncertain Objective Function

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Abstract

In this paper we analyze a class of multi-parametric quadratic program (mpQP) with parameters in the objective function. Except for parameters in coefficients associated with the linear term, the coefficient of the quadratic term, which is a positive definite matrix, is multiplied by a scalar parameter, while the quadratic coefficient of a standard mpQP is deterministic. We reveal the optimal solution is a linear fractional function in the parameters, and the critical regions remain polyhedral. The discussed mpQP can be reformulated as a standard mpQP via variable and parameter transformations. The proposed method is used to evaluate the economic operation of a residential energy system under time-and-level-of-use electricity pricing, highlighting the potential application in practical problems.

Keywords:
multi-parametric quadratic program, critical regions, objective uncertainty, residential energy system

1. Introduction

We considering the following multi-parametric quadratic program (mpQP):

$$\begin{aligned} &\min \ \frac{1}{2}\sigma x^\top Q x + c(\theta)^\top x \\ &\text{s.t.} \ \ Ax \le b \\ &\Theta = \left\{ (\sigma,\theta) \in \mathbb{R}^{p+1} \;\middle|\; \begin{matrix} \sigma_l \le \sigma \le \sigma_m \\ \theta_l \le \theta \le \theta_m \end{matrix} \right\} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables; $\theta \in \mathbb{R}^p$ is the vector of parameters; $c \in \mathbb{R}^n$ is the coefficient vector associated with the linear term, and its elements are linear functions in θ, i.e., $c(\theta) = c_0 + C\theta$, where $c_0 \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n\times p}$ are constants; $Q \succ 0$ is an $n \times n$ positive definite matrix; without loss of generality, we assume Q is symmetric; otherwise, because $x^\top Q x = x^\top Q^\top x$, we can always replace Q by $(Q + Q^\top)/2$ without changing the objective function, while the latter matrix is symmetric; $\sigma \in \mathbb{R}_+$ is a non-negative scalar parameter corresponding to the quadratic term. Θ is the set of admissble parameters, which is assumed to be a hypercube defined by lower bound σ_l/θ_l and upper bound σ_m/θ_m; in more general cases, Θ could be a full-dimensional polyhedron. $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are constant coefficients of linear inequality constraints. Equality constraints can be eliminated via expressing some variables using remaining ones and substituting them into inequalities. Notice that replacing an equality with two opposite inequalities may not be a good choice, because the coefficients of the two constraints are linearly dependent, and they become active at the same time, which may ruin the linear independence constraint qualification. Variable elimination overcomes this potential pitfall, and also reduces problem size.

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When $\sigma > 0$ is fixed, the coefficient of the quadratic term is deterministic. In such a circumstance, problem (1) is a standard mpQP which has been well-studied in the existing literature, such as in Tøndel et al. (2003a); Bemporad (2015); Dua et al. (2002); Pistikopoulos et al. (2007); Gupta et al. (2011). In a standard mpQP, the parameter θ may appear in the constraint right-hand side and coefficients of the linear term in the objective function. It is known that the optimal solution of a standard mpQP is piecewise affine in θ, and hence the optimal value is a piecewise quadratic function in θ. When $\sigma = 0$, mpQP (1) comes down to a multi-parametric linear program with objective uncertainty, which has been studied in Hadigheh & Terlaky (2006) and more generally in Charitopoulos et al. (2017). For any given θ, the dual of the linear program in (1) is

$$\begin{aligned} \min\ & b^\top \mu \\ \text{s.t.}\ & A^\top \mu = c(\theta) \\ & \mu \leq 0 \end{aligned} \tag{2}$$

where μ is the vector of dual variables. Consider all $\theta \in \Theta$, the optimal value is a function of θ, and problem (2) appears to be a standard multi-parametric linear program and can be solved by the method in Borrelli et al. (2003); Hladík (2010). In the rest of this paper, we only consider strictly positive parameter σ in the quadratic term, and thus the quadratic program is strictly convex.

2. Critical Regions and the Optimal Value

Suppose problem (1) is solved for given parameters $\sigma > 0$ and θ. Let matrix $\bar{A}$ and $\bar{b}$ denote the rows of A and b associated with active constraints, and $\hat{A}$ and $\hat{b}$ correspond to the coefficients of remaining constraints which are inactive. $\bar{\mu}$ and $\hat{\mu}$ are dual variables corresponding to active and inactive constraints, respectively. Since problem (1) is strictly convex, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for optimality, and the primal and dual variables must satisfy

$$\sigma Q x + c(\theta) + \bar{A}^\top \bar{\mu} = 0 \tag{3a}$$

$$\bar{A} x = \bar{b} \tag{3b}$$

$$\hat{A} x < \hat{b} \tag{3c}$$

$$\bar{\mu} \geq 0 \tag{3d}$$

$$\hat{\mu} = 0 \tag{3e}$$

Then the primal optimal solution can be solved from (3a) as

$$x = -\sigma^{-1} Q^{-1} \left[c(\theta) + \bar{A}^\top \bar{\mu} \right] \tag{4}$$

Substituting (4) into equality (3b), we have

$$\bar{A} Q^{-1} \left[c(\theta) + \bar{A}^\top \bar{\mu} \right] = -\bar{b}\sigma \tag{5}$$

If $\bar{A}$ has full-rank in its rows, then matrix $\bar{A} Q^{-1} \bar{A}^\top \succ 0$ and hence invertible. So the dual optimal solution can be solved as

$$\bar{\mu} = -Q_{\bar{A}}^{-1} \left[\bar{A} Q^{-1} c(\theta) + \bar{b}\sigma \right] \tag{6}$$

where $Q_{\bar{A}} = \bar{A} Q^{-1} \bar{A}^\top$. Further substituting (6) in (4) gives the primal optimal solution

$$x = -\sigma^{-1} Q^{-1} \bar{M} c(\theta) + Q^{-1} \bar{N} \tag{7}$$

where $\bar{M} = I - \bar{A}^\top Q_{\bar{A}}^{-1} \bar{A} Q^{-1}$, $\bar{N} = \bar{A}^\top Q_{\bar{A}}^{-1} \bar{b}$.

From (6) and (7), although the dual optimal solution is a linear function in parameters (σ, θ) in some certain region, the primal solution is nonlinear in σ, so is the optimal value. More precisely, x is a linear fractional function in σ and θ, and the denominator is σ. This is a generalization of standard mpQPs with deterministic σ, whose optimizer is an affine function of θ in each critical region.

As long as the set of active constraints does not change, the optimal primal solution x and dual solution μ can be expressed by (7) and (6), respectively, regardless of the values of parameters. The set of (σ, θ) for x in (7) being optimal is called the critical region. To obtain the critical region, substituting (6) and (7) in (3d) and (3c), respectively, the pair of parameters (σ, θ) must satisfy

$$\begin{aligned} -Q_{\bar{A}}^{-1} \left[\bar{A} Q^{-1} c(\theta) + \bar{b}\sigma\right] \geq 0 \\ \hat{A}\left(-\sigma^{-1} Q^{-1} \bar{M} c(\theta) + Q^{-1} \bar{N}\right) < \hat{b} \end{aligned} \tag{8}$$

So the critical region is

$$\mathrm{CR} = \left\{ (\sigma, \theta) \,\middle|\, \begin{array}{c} -Q_{\bar{A}}^{-1} \left[\bar{A} Q^{-1} c(\theta) + \bar{b}\sigma\right] \geq 0 \\ \left(\hat{A} Q^{-1} \bar{N} - \hat{b}\right) \sigma < \hat{A} Q^{-1} \bar{M} c(\theta) \end{array} \right\} \tag{9}$$

Clearly, the closure of a critical region is polyhedral. Any parameter in a certain critical region yields the same set of active constraints at optimum. Once a critical region has been identified, (7) is an analytical expression of the optimal solution. Moreover, if Θ is divided into the union of critical regions, solving quadratic program in (1) boils down to a lookup table of matching the parameter with critical regions.

3. Using an mpQP solver via Transformation

Given the expression in (9), critical regions can be drawn by visiting different values of $\theta \in \Theta$. In case that they are overlapping, the parameter set partitioning method in Tøndel et al. (2003b); Borrelli et al. (2003) can be applied. Nonetheless, several mature solvers for multi-parametric programming problems have been developed, such as the multi-parametric toolbox (MPT) Kvasnica et al. (2004) and the parametric optimization toolbox (POP) Oberdieck et al. (2016). In this section, we discuss a variable and parameter transformation method. It reformulates problem (1) as a standard mpQP which can be processed by existing mpQP solvers.

Consider the following mpQP

$$\begin{aligned} \min \ & \frac{1}{2} x^\top Q x + \frac{c(\theta)^\top x}{\sigma} + \frac{c(\theta)^\top Q^{-1} c(\theta)}{2\sigma^2} \\ \text{s.t.} \ & Ax \leq b, (\sigma, \theta) \in \Theta \end{aligned} \tag{10}$$

where Q^{-1} is the inverse of Q. Problems (10) and (1) share the same solution, as an affine mapping of the objective function does not affect the optimal solution. Define a new variable z via transformation

$$x = z - \frac{Q^{-1} c(\theta)}{\sigma} \tag{11}$$

Problem (10) comes down to a new mpQP

$$\begin{aligned} \min \ & \frac{1}{2} z^\top Q z \\ \text{s.t.} \ & Az \leq b + \frac{A Q^{-1} c(\theta)}{\sigma} \\ & (\sigma, \theta) \in \Theta \end{aligned} \tag{12}$$

via the new variable z.

Furthermore, define a parameter transformation

$$\sigma' = \frac{1}{\sigma} \in \mathbb{R}_{++}, \quad \theta' = \frac{\theta}{\sigma} \in \mathbb{R}^p \tag{13}$$

Then the new parameter set becomes

$$\Theta' = \left\{ (\sigma', \theta') \;\middle|\; \begin{array}{c} \sigma_m^{-1} \le \sigma' \le \sigma_l^{-1} \\ \theta_l/\sigma_m \le \theta' \le \theta_m/\sigma_l \end{array} \right\} \tag{14}$$

Substituting $c(\theta) = c_0 + C\theta$ in (12) and replacing parameters σ and θ with new ones in (13), we have the following mpQP with a solver-compatible form:

$$\begin{aligned} &\min \ \frac{1}{2} z^\top Q z \\ &\text{s.t.} \quad Az \le b + AQ^{-1}[c_0, C] \begin{bmatrix} \sigma' \\ \theta' \end{bmatrix} \\ &\qquad (\sigma', \theta') \in \Theta' \end{aligned} \tag{15}$$

According to the theory of mpQP, the critical regions of (15) is polyhedral, for example

$$D \begin{bmatrix} \sigma' \\ \theta' \end{bmatrix} \le d \tag{16}$$

Apply inverse parameter transformation, the critical regions in the original parameters is

$$d\sigma - D_r\theta \ge D_1 \tag{17}$$

where D_1 is the first column of D, and D_r is the remaining part of D except D_1. It remains a polyhedron, which is consistent with the analysis in Section 2. The optimal solution z in a certain critical region is a linear function in (σ', θ'). Applying variable and parameter transformations in (11) and (13), we can express x as piecewise linear fractional functions in (σ, θ); consequently, the optimal value function can be obtained.

Remark: When mapped back to the original parameter space, the image of the new parameter set Θ' is larger than the original parameter set Θ. Alternatively, because

$$\sigma = \frac{1}{\sigma'}, \ \theta = \theta'\sigma = \frac{\theta'}{\sigma'} \tag{18}$$

the new parameter set can be defined as

$$\Theta' = \left\{ (\sigma', \theta') \;\middle|\; \begin{array}{c} \sigma_m^{-1} \le \sigma' \le \sigma_l^{-1} \\ \sigma'\theta_l \le \theta' \le \sigma'\theta_m \end{array} \right\} \tag{19}$$

It is no longer a hypercube, but the intersection of a cone and a slab, and thus remains polyhedral. Its image on the original parameter space is Θ. Therefore, the exploration of unnecessary region in the new parameter space can be circumvented.

4. Economic Assessment of a Residential Energy System

We consider a residential energy system with an ideal energy storage which is lossless. System configuration is shown in Fig. 1. The solar panel output, storage charging and discharging power, grid power supply and load in periods $t = 0, 1, \cdots, 23$ are denoted by p_t^V, p_t^c, p_t^d,p_t^n and l_t, respectively. Load and solar power forecasts are used in

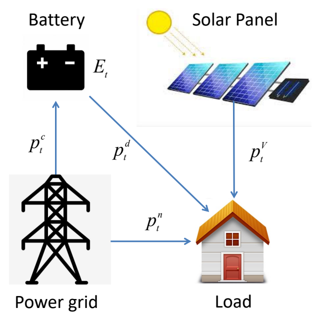

Figure 1: Configuration of the Residential Energy System

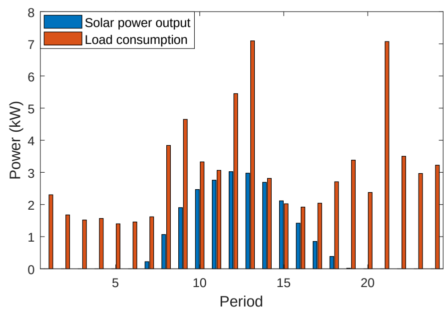

Figure 2: Curves of Load and Solar Power Output

the model, so P_t^V and l_t are input data, which have been provided in Fig. 2; p_t^c, p_t^d, and p_t^n are to be determined by the energy management system.

According to the energy flows in Fig. 1, balancing condition

$$l_t = p_t^n + p_t^V + p_t^d, \forall t \tag{20}$$

holds in period t; the total power bought from the power grid is

$$p_t^e = p_t^c + p_t^n, \forall t \tag{21}$$

The time-and-level-of-use electricity price Duarte et al. (2020) is an affine function in the power consumption

$$\lambda_t = \tilde{\lambda}_t + k p_t^e / 2, \tag{22}$$

The base price is

$$\tilde{\lambda}_t = \begin{cases} 5.0 \ \ \text{cent/kWh}, & t \in T_L \\ \lambda_h \ \ \text{cent/kWh}, & t \in T_H \end{cases} \tag{23}$$

where T_H represents peak periods which last from 10:00 a.m. to 9.00 p.m., and T_L includes the remaining periods of the day. k and λ_h are parameters and their impact on the total cost will be investigated. The power-dependent term in (22) can prevent rebounds peak when the based price drops down.

The operation of energy storage unit satisfies

$$0 \le p_t^c \le p_m, \ 0 \le p_t^d \le p_m, \forall t \tag{24a}$$

$$E_{t+1} = E_t + (p_t^c - p_t^d)\Delta_t, \forall t \tag{24b}$$

$$0.2E_m \le E_t \le E_m, E_0 = 0.2E_m, \forall t \tag{24c}$$

where p_m = 5kW is the maximum charging/discharging power; E_m = 30kWh is the capacity of the storage unit. The default duration of each period is $\Delta_t = 1$. (24a) restricts the maximum charging and discharging power; (24b) describes the dynamics of state-of-charge (SoC); since the storage is ideal, the efficiency is equal to 1, and the loss during charging/discharging is neglected; (24c) stipulates the energy capacity and initial SoC; a lower bound is also imposed on the storage SoC, preventing a deep discharge which may injure the storage facility.

Eliminating p_t^n from (20) and (21) we have

$$p_t^e = p_t^c - p_t^d + l_t - p_t^V, \forall t \tag{25}$$

Define a new variable $p_t^s = p_t^c - p_t^d$ for the energy storage, (25) can be written as

$$p_t^s = p_t^e + p_t^V - l_t, \forall t \tag{26}$$

and (24b) becomes

$$E_{t+1} = E_t + (p_t^e + p_t^V - l_t)\Delta_t, \forall t \tag{27}$$

Finally, eliminating E_t and equality constraint (27) yields

$$0.2E_m \le E_0 + \sum_{j=1}^{t}(p_j^e + p_j^V - l_j)\Delta_j \le E_m, \forall t \tag{28}$$

Now, all constraints become linear inequalities.

Remark: Constraint (26) naturally suggests that if the solar output is larger than the load, the excessive power is used to charge the storage unit, although such a connection is not explicitly shown in Fig. 1.

In summary, the economic operation of the residential energy system gives rise to the following mpQP:

$$\begin{aligned}
\min \quad & 5\sum_{t\in T_L} p_t^e\Delta_t + \lambda_h \sum_{t\in T_H} p_t^e\Delta_t + \frac{k}{2}\sum_t \left(p_t^e\right)^2\Delta_t \\
\text{s.t.} \quad & -p_m \le p_t^e + p_t^V - l_t \le p_m, \forall t \\
& 0.2E_m \le E_0 + \sum_{j=1}^{t}(p_j^e + p_j^V - l_j)\Delta_j \le E_m, \forall t \\
& \Theta = \{(k,\lambda_h) \mid 0.1 \le k \le 4, \quad 8 \le \lambda_h \le 12\}
\end{aligned} \tag{29}$$

In problem (29), decision variable is p_t^e; p_t^V and l_t are retrieved from forecast; parameters k and λ_h correspond to the quadratic term and the linear term, respectively. Problem (29) has the same form as the mpQP in (1).

4.1. Only k is uncertain

First, we consider the case in which $\lambda_h = 10$ cent is fixed. The analysis in Section 2 is used to compute the optimal value function and critical regions, which are actually intervals in this case. Foremost, problem (29) is solved for $k = 0.1$, and the active constraints are picked out. Then the corresponding critical interval and the optimal solution in this interval can be computed via (9) and (7), respectively. Afterwards, problem (29) is solved again with a new k outside the obtained interval, and a new critical interval and optimal solution can be retrieved. The procedure continues until Θ are covered by critical intervals. Finally, Θ is divided into 4 critical intervals, and the optimal value function is given by

$$v(k) = \begin{cases} 258.31 + 110.62k, & \text{if } 0.10 \le k \le 1.07 \\ 374.01 + 56.54k - 61.87/k, & \text{if } 1.07 \le k \le 1.79 \\ 378.03 + 55.41k - 65.48/k, & \text{if } 1.79 \le k \le 2.48 \\ 383.08 + 54.40k - 71.74/k, & \text{if } 2.48 \le k \le 5.00 \end{cases} \tag{30}$$

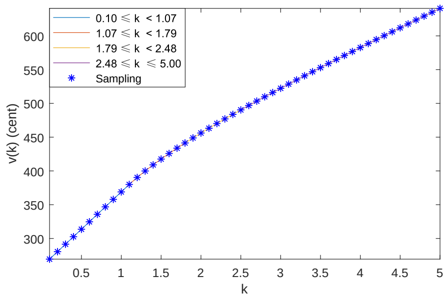

Figure 3: Optimal value function

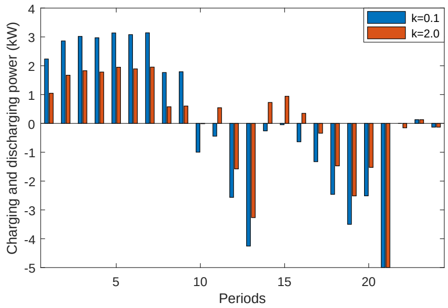

Figure 4: Storage operation strategy under different k

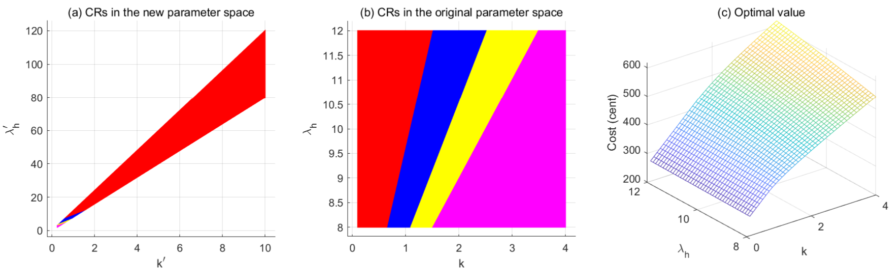

Figure 5: Results of economic assessment on the residential energy system

The optimal value function is plotted in Fig. 3. To validate its accuracy, problem (29) is solved at discrete samples of k with a step size of 0.1; the respective optimal values are marked in the same figure. It is observed that $v(k)$ offered by (30) exactly matches those obtained from solving problem (29) directly.

Storage operation strategies under $k = 0.1$ and $k = 2$ are compared in Fig. 4. With a larger value of k, the electricity price grows more quickly with respect to load. Therefore, less power is purchased from the power grid during valley periods to charge the energy storage unit. As a result, the consumer has to buy more power during peak hours, leading to a higher daily operation cost.

4.2. Both k and λ_h are uncertain

Now, we solve problem (29) with both uncertain parameters. It is transformed to a traditional mpQP in the form of (15), and then POP toolbox Oberdieck et al. (2016) is called. Results are shown in Fig. 5. Four critical regions in $\mathbb{R}^2$ are found and portrayed with new parameters in Fig. 5a. The ranges of k' and λ'_h are defined by (19). Mapping back to the original parameter space, the critical regions are displayed in Fig. 5b. From Fig. 5c we can see that k affects the daily operation cost more significantly compared to λ_h. For a fixed k, the cost grows with the increase of λ_h, and the growing rate becomes larger when the value of k increases. We also observe that even if k is allowed to take values larger then 4, no more critical regions will be created, which means the set of active constraints will not change if k exceeds a certain value, so does the optimal operation strategy, as the marginal cost induced by the quadratic term overwhelms the gap between the peak price and valley price.

5. Conclusions

In this paper, we discuss a class of mpQP with uncertain parameters in the objective function, where the quadratic term is multiplied by a scalar parameter. The optimal solution is characterized by a linear fractional function in parameters, and the critical region is shown to be polyhedral. The connection with standard mpQP is revealed via variable and parameter transformation, which allows the use of off-the-shelf mpQP solvers.

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