Maximum Loadability of Meshed Networks: A Sequential Convex Optimization Approach

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***Abstract*—In power system static security analysis, it often requires to calculate continuous power flow from a certain load condition to a bifurcation point along a given direction, which is referred to as the maximum loadability problem. This paper proposes a convex optimization method for maximum loadability problem over meshed power grids based on the semidefinite relaxation approach. Because the objective is to maximize the load increasing distance, convex relaxation model is generally inexact, unlike the situation in cost-minimum optimal power flow problem. Inspired by the rank penalty method, this paper proposes an iterative procedure to retrieve the maximum loadability. The convex quadratic term representing the penalty on the rank of matrix variable is updated in each iteration based on the latest solution. In order to expedite convergence, generator reactive power is also included in the objective function, which has been reported in literature. Numeric tests on some small-scale systems validate its effectiveness. Any sparsity-exploration and acceleration techniques for semidefinite programming can improve the efficiency of the proposed approach.**

***Index Terms*—maximum loadability, meshed network, convex optimization, semidefinite programming**

NOMENCLATURE

Frequently used symbols and notations are defined here for quick reference. Others are clarified following their first appearance as required.

A. Parameters

G_{ij}	Conductance between bus i and j
B_{ij}	Admittance between bus i and j
V_{im}^2	Upper limit of voltage amplitude at bus i
V_{il}^2	Lower limit of voltage amplitude at bus i
p_{im}^g	Upper limit of generation active power at bus i
p_{il}^g	Lower limit of generation active power at bus i
q_{im}^g	Upper limit of generation reactive power at bus i
q_{il}^g	Lower limit of generation reactive power at bus i
F_{ij}^l	Flow limit of transmission line l connecting buses i and j
r_{ij}^l, x_{ij}^l	Resistance and reactance of line l.
p^{d0}, q^{d0}	Initial active and reactive power demand

B. Variables

p_i^g	Active power generation at bus i.
q_i^g	Reactive power generation at bus i.
p_i^d	Active power demand at bus i.
q_i^d	Reactive power demand at bus i.
e_i, f_i	Real and imaginary part of complex voltage of bus i in the rectangular coordinate

I. INTRODUCTION

A. Motivation and Background

With the growth of economy and power demand, modern power systems are often operated under stressed conditions, raising the risk of voltage collapse. Static voltage stability of a power system is closely related to power flow solvability. Since the nodal load demand is the parameter of power flow routine, it is natural to ask how much load a power system can carry before power flow equations become insolvable. In this regard, a basic task is to examine the power flow solution while increasing the system load along a direction, until no solution exists. The corresponding distance reflects the maximum loadability (MLA) of the power system associated with the initial point and load increasing pattern, and offers grid operator an intuitive index of security margin. However, solving power flow equations near the boundary of infeasibility encounters numeric issues due to the high condition number. Special technique is needed to overcome this difficulty.

B. Relevant Literature

The major computation methods for MLA can be categorized into two types. One is the continuation power flow

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(CPF) method, and the other is optimization based approach. CPF method starts from the current operation point, and repeatedly solves power flow equations with the step increase of load [1]–[4]. Around the voltage collapse point, special prediction/correction methods are designed to avoid numeric instability. Optimization based method computes the MLA by solving a special optimal power flow problem which maximizes the length of load increment subject to power flow equations and other variable bound constraints [5]–[7]. According to [6], compared with CPF method, optimization based approach is more flexible to deal with various operating constraints, and takes advantage of off-the-shelf solvers.

Although CPF and optimization based methods are well developed and have been applied in large-scale systems, some issues remain. On the one hand, although the CPF method is efficient, it encounters difficulty when constraints are considered, such as power flow limits of transmission lines, because the search direction is tightly coupled with the set of binding constraints. However, before the problem is solved, it is not clear which constraint will be binding. On the other hand, although optimization method can deal with various constraints, MLA gives rise to a non-convex program, and the computational performance of general-purpose nonlinear programming solvers may be sensitive to system parameter, depending on the algorithm used. Even for a small-scale system, a solver may fail to report a solution, given the tough structure near a bifurcation point.

Since the pioneer work in [8], [9], convex relaxation method is shown to be promising in optimal power flow problem, including semidefinite programming (SDP) relaxation [8] and second-order cone programming (SOCP) relaxation [9]. Convex relaxation has been used in the MLA problem. In [10], the SDP relaxation is employed to construct sufficient conditions for power flow insolvability and approximate voltage stability margins. Because SOCP is more tractable from the computational perspective, the SOCP relaxation is used to calculate voltage stability margins in [11]. Unlike the cost-minimum optimal power flow problem, in which convex relaxation is exact under come mild conditions, there is no guarantee that convex relaxation will be exact in MLA or voltage stability problem. In fact, the optimality gap of the SOCP relaxation may be very large. In a recent study [12], a sufficient condition for non-singularity of power flow Jacobian is derived and formulated as second-order cones, which is further combined with the SOCP relaxation of optimal power flow.

As mentioned above, when applied in the MLA problem, convex relaxations are generally inexact, and hence the obtained optimal solution is optimistic, or in other words, the system will get crashed in practice before the load can reach the indicated level. On this account, tightening the convex relaxation, at least estimating a gap of the relaxed solution, is also important in practical usage.

C. Contributions and Organization

In this paper, we propose a sequential SDP method to solve the MLA problem. It starts with the renowned SDP relaxation (SDR); if the result is inexact, which can be easily examined by checking the singular values of the solution matrix, rank penalty term is added in the objective function and the new problem is solved again. In addition to the fixed penalty term that associated with generator reactive power proposed in [13], [14], the proposed approach exploits a variable penalty term, which guarantees to find a feasible solution. Compared to the CPF method, it inherits the advantage of optimization based formulation; furthermore, it leverages the computational superiority of convex optimization, compared to the traditional ones that directly handles non-convex formulations. Unlike a local nonlinear programming solver that needs an initial guess, the start point of the penalty model is offered by the SDP relaxation, while the SDP solver makes no reference to a manually supplied initiation. Finally, a sequential SOCP method is proposed in [15] to cope with general power flow optimization problems in distribution networks. The power flow model therein is established based on branch flow equations, which is only suitable for radial power grids. In this paper, the most common bus injection power flow model (BIM) is used, which is valid for both meshed and radial networks. The penalty terms also exhibit different meanings because the SOCP model has no rank information. Compared to the earlier studies, we adopt penalty functions in the MLA optimization problem and output the result according to the algorithm convergence criteria, thus ensuring the feasibility and correctness of the final solution. A possible shortcoming might be the efficiency, as the SDP model uses a square (not vector) matrix variable. Nonetheless, any sparsity-exploration and acceleration techniques for SDP can improve the efficiency of the proposed approach.

The rest of this paper is organized as follows. The basic formulation and compact form of the MLA problem are presented in Section II. The sequential SDP algorithm is also developed in this section. The algorithm is compared with some popular solvers in case study, and results are reported in Section III. Finally, conclusions are drawn in Section IV.

II. Maximum Loadability Problem

The MLA problem aims to maximize the load increasing distance subject to power flow equations are other system operation constraints. Most symbols and notations will be borrowed from [8].

A. Mathematical Formulation

The power grid is modeled by a graph $N = (B, L)$, where B denotes the set of buses; its cardinality is $|B| = n$. Without loss of generality, bus n is chosen as the slack bus; L denotes the set of lines. The nodal admittance matrix is $Y \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^{n\times n}$ is the set of $n \times n$ complex matrix, and its elements is $Y_{ij} = G_{ij} + \mathrm{i}B_{ij}$, $\forall i, j \in B$, where $\mathrm{i} = \sqrt{-1}$. Let p_i^g and q_i^g be the active and reactive output of generator at bus i ($p_i^g = q_i^g = 0$ if there is no generator at that bus), respectively. Let p_i^d and q_i^d be active and reactive power consumptions at bus i ($p_i^d = q_i^d = 0$ if there is no load at that bus), respectively. The complex voltage of bus i in the rectangular coordinate can be expressed as $V_i = e_i + \mathrm{i} f_i$. With above notations, power flow equations can be expressed as follows

$$\begin{aligned} p_i^g - p_i^d = &\sum_{j=1}^{n} G_{ij}(e_i e_j + f_i f_j) \\ &- \sum_{j=1}^{n} B_{ij}(e_i f_j - e_j f_i), \quad \forall i \in B \end{aligned} \tag{1}$$

$$\begin{aligned} q_i^g - q_i^d = &- \sum_{j=1}^{n} B_{ij}(e_i e_j + f_i f_j) \\ &- \sum_{j=1}^{n} G_{ij}(e_i f_j - e_j f_i), \quad \forall i \in B \end{aligned} \tag{2}$$

$$e_i^2 + f_i^2 - V_{im}^2 \le 0, \ \forall i \in B \tag{3}$$

$$V_{il}^2 - e_i^2 - f_i^2 \le 0, \ \forall i \in B \tag{4}$$

$$V_{nl} \le e_n \le V_{nm}, \ f_n = 0 \tag{5}$$

$$p_{il}^g \le p_i^g \le p_{im}^g, \ q_{il}^g \le q_i^g \le q_{im}^g, \ \forall i \in B \tag{6}$$

$$\frac{(e_i - e_j)^2 + (f_i - f_j)^2}{r_{ij}^2 + x_{ij}^2} \le (F_{ij}^l)^2, \ \forall l \in L \tag{7}$$

where equalities (1) and (2) are nodal active and reactive power balancing conditions in the rectangular coordinate; constraints (3) and (4) restrict bus voltage magnitude within the interval $[V_{il}, V_{im}]$; (5) enforces the voltage at the slack bus being a real number; (6) is generator capacity constraint, where p_{il}^g and q_{il}^g are the minimal active and reactive output, respectively; p_{im}^g and q_{im}^g are the maximal active and reactive output, respectively. (7) imposes current flow limit F_{ij}^l if transmission line l connecting buses i and j. The reason we choose current restriction instead of power restriction is that the thermal limit of a transmission line mainly depends on the current it carries, and the current at the head and tail buses are equal, while inflow and outflow of a line can be different due to losses. In fact, different line flow limit constraints lead to minor differences. In the MLA problem, system load grows from an specified point p_i^{d0}, q_i^{d0}, $\forall i$ along direction Δp, Δq, and thus can be expressed by

$$\begin{aligned} p^d &= p^{d0} + \lambda \Delta p \\ q^d &= q^{d0} + \lambda \Delta q \end{aligned} \tag{8}$$

where scaler variable λ indicates the distance moved from the known operating point along the given direction. In summary, the MLA problem can be presented as

$$\max\{\lambda \mid (1) - (8)\} \tag{9}$$

B. SDP Relaxation Model

Problem (9) is non-convex due to quadratic equalities (1) and (2) as well as non-convex quadratic inequality (4). SDP relaxation can be applied. In SDP relaxation model, we use the following notations, which have been devised in [8]. Vectors b_1, b_2, $\cdots$, b_n denote the i-th column of an $n \times n$ identity matrix. Vectors $d_{ij}^l = b_i - b_j$, $\forall l \in L$. Voltage real part vector is $e = [e_1, e_2, \cdots, e_n]$, and voltage imaginary part vector is $f = [f_1, f_2, \cdots, f_n]$. $x = [e^T, f^T]^T$, and matrices

$$Y_k = b_k b_k^T Y \in \mathbb{C}^{n \times n}$$

$$M_k = \begin{bmatrix} b_k b_k^T & 0 \\ 0 & b_k b_k^T \end{bmatrix}, N_l = \begin{bmatrix} d_{ij}^l d_{ij}^{lT} & 0 \\ 0 & d_{ij}^l d_{ij}^{lT} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

$$Z_k = \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

$$\bar{Z}_k = -\frac{1}{2} \begin{bmatrix} \mathrm{Im}\{Y_k + Y_k^T\} & \mathrm{Re}\{Y_k - Y_k^T\} \\ \mathrm{Re}\{Y_k^T - Y_k\} & \mathrm{Im}\{Y_k + Y_k^T\} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

where operators $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ represent the element-wise real and imaginary part of a complex vector or matrix. With the help of the above notations, the compact form of constraints (1), (2), (3), (4) and (7) can be represented by

$$x^T Z_k x = P_{in,k}(\lambda), \ \forall k \in B \tag{10}$$

$$x^T \bar{Z}_k x = Q_{in,k}(\lambda), \ \forall k \in B \tag{11}$$

$$V_{kl}^2 \le x^T M_k x \le V_{km}^2, \ \forall k \in B \tag{12}$$

$$x^T N_l x \le (z_{ij}^l F_{ij}^l)^2, \ \forall l \in L \tag{13}$$

where scalars $P_{in,k}(\lambda) = p_k^g - p_k^d$ and $Q_{in,k}(\lambda) = q_k^g - q_k^d$ are active and reactive power injections, respectively, and they implicitly depends on λ following the linear relationship (8). Constraints (12) include constraints (3) and (4), and in (13), $(z_{ij}^l)^2 = (r_{ij}^l)^2 + (x_{ij}^l)^2$ is the impedance of line l. Because quadratic form has the following transformation

$$x^T Q x = \mathrm{Tr}(x^T Q x) = \mathrm{Tr}(Q x x^T)$$

where $\mathrm{Tr}(\cdot)$ is the matrix trace operator. Define matrix variable $X = xx^T \in \mathbb{R}^{n \times n}$, then (10)-(13) can be linearized as

$$\mathrm{Tr}(Z_k X) = P_{in,k}, \ \forall k \in B \tag{14}$$

$$\mathrm{Tr}(\bar{Z}_k X) = Q_{in,k}, \ \forall k \in B \tag{15}$$

$$V_{kl}^2 \le \mathrm{Tr}(M_k X) \le V_{km}^2, \ \forall k \in B \tag{16}$$

$$\mathrm{Tr}(N_l X) \le (z_{ij}^l F_{ij}^l)^2, \ \forall l \in L \tag{17}$$

and problem (9) can be equivalently reformulated as

$$\max\{\lambda \mid (5) - (6), (8), (14) - (17), \ \mathrm{Rank}(X) = 1\} \tag{18}$$

The last constraint ensures the feasibility of rank-1 decomposition $X = xx^T$, which means a meaningful power flow status can be recovered. However, rank-1 constraint remains non-convex. To remove non-convexity, it is replaced with a linear matrix inequality, and the SDP relaxation model reads

$$\max\{\lambda \mid (5) - (6), (8), (14) - (17), X \succeq 0\} \tag{19}$$

SDP (19) can be solved by conic programming solvers such as MOSEK, SDPT3 and SEDUMI. If the optimal solution of problem (19) is rank-1, satisfying the rank constraint in (18), then the decomposition $X = xx^T$ recovers the optimal solution x of problem (9). However, in most cases, the relaxed solution has a rank higher than 1, and the corresponding λ is only an upper bound of the true maximum loadability. Hence, further methods are required to attain a rank-1 solution.

C. The Sequential SDP Algorithm

In order to obtain a rank-1 solution, a sequential SDP algorithm developed in [16] is applied, due to its good convergence and computational efficiency. In MLA problem (18), we define the following two sets

$$
\begin{aligned}
C_X &= \{X \in \mathbb{S}^n \mid X \succeq 0, (5)-(6), (8), (14)-(17)\} \\
D_Y &= \{Y \in \mathbb{S}^n \mid \text{Rank}(Y) = 1\}
\end{aligned}
$$

Consider the following problem

$$
\begin{aligned}
\max \quad & \lambda - \frac{\alpha}{2}\|X - Y\|^2 \\
\text{s.t.} \quad & X \in C_X, Y \in D_Y
\end{aligned}
\tag{20}
$$

where α is a positive penalty parameter, and $\|\cdot\|$ denotes the Frobenius norm. If the optimal solution leads to a zero penalty, i.e., $X = Y$, then X must be rank-1. Although problem (20) is non-convex due to the existence of D_Y, the constraints imposed on X and Y are decoupled. This fact motivates alternate optimization of X and Y until a convergence tolerance is met.

To this end, we consider the rank-1 constrained problem

$$
\begin{aligned}
\min \quad & \frac{\alpha}{2}\|X_k - Y\|^2 \\
\text{s.t.} \quad & Y \in D_Y
\end{aligned}
\tag{21}
$$

where X_k is constant. Problem (21) gives the best rank-1 approximation of X_k. Since X_k can be written as

$$
X_k = \sum_{i=1}^{r} \eta_i p_i p_i^T \tag{22}
$$

where $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_r \geq 0$ are the eigenvalues of square matrix X and $p_1, \cdots, p_r \in \mathbb{R}^n$ are the corresponding eigenvectors. Clearly, $\eta_1 p_1 p_1^T$ is the best rank-1 approximation [17], [18].

When $Y = Y_k$ is fixed, problem (20) comes down to the SDP relaxation model with a rank penalty in the objective function. In order to expedite convergence, the penalty terms proposed in [13], [14], which are represented by reactive power of generators, are also added into the objective function. Hence we solve SDP

$$
\max \left\{ \lambda - \frac{\alpha}{2}\|X - Y_k\|^2 - \beta \sum q_i^g \mid X \in C_X \right\} \tag{23}
$$

The sequential SDP algorithm is summarized as follows.

In Algorithm 1, we do not use a constant penalty parameter α; instead, we increase its value from a small number, and limit its largest value. The reason is: a small penalty at the beginning helps locate the global optimal solution. Otherwise, although a feasible rank-1 solution can be found quickly, the optimality might be sacrificed. However, if the penalty parameter goes too large, the problem will suffer from a high conditional number, which brings numeric problems and does not facilitate computation.

Algorithm 1 Sequential SDP for MLA

1: Select a convergence tolerance $\varepsilon > 0$, penalty coefficients $\alpha_0 > 0$, $\beta > 0$, and penalty growth rate $\tau > 1$, the maximum penalty coefficient $\alpha_{max} > 0$. Solve SDP relaxation model (19). The optimal solution is X^*, and the iteration index is $k = 0$.

2: Apply singular value decomposition to X_k, sort the singular values $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0$ in a descent order. If $r = 1$, X is exactly rank-1. In practice, we can accept an approximate solution if $\mu_2, \cdots, \mu_r$ are small enough. Compute

$$
R = \frac{\sum_{i=2}^{r} \mu_i}{\mu_1}
$$

if $R < \varepsilon$, terminate and report the optimal solution.

3: Let $Y_k = \eta_1 p_1 p_1^T$; update $\alpha_{k+1} = \min\{\tau\alpha_k, \alpha_{max}\}$, $k = k+1$, and solve problem (23), the optimal solution is X_k. Go to step 2.

TABLE I
COMPUTATION PERFORMANCE OF ALGORITHM 1

System	λ	Time(s)	Iter
case6	1.0842	2.4211	0
case9	2.2892	2.2724	0
case30	1.0353	5.9049	1
case39	1.0903	12.9522	1

III. CASE STUDY

To verify the effectiveness of Algorithm 1, IEEE benchmark 6-bus, 9-bus, 30-bus and 39-bus systems are tested. All numeric experiments are performed on a laptop with Intel i5-7267U CPU and 8GB memory. Optimization problems are coded in MATLAB with YALMIP interface [19], and solved by MOSEK [20]. For comparison, the MLA problem (9) in its original form is solved by general-purpose nonlinear programming solvers, including global solvers BARON which adopts branch-and-bound type method and SCIP. We also tried IPOPT applying interior point method and SNOPT utilizing sparse SQP (Sequential Quadratic Program) method, but they failed to solve the problem. Since reactive power limits lead the primal problem in SDR to be a mixed-integer one [21] (due to bus type switch), reactive power limits are neglected in our tests, which is also adopted in [11]. Computation results are shown in Table I and Table II.

TABLE II
COMPUTATION PERFORMANCE OF NLP SOLVERS

	BARON		SCIP	
System	λ	Time(s)	λ	Time(s)
case6	1.0846	7.6950	1.0842	6.7736
case9	2.2892	6.2029	2.2892	54.1902
case30	1.0383	1011	Fail	-
case39	1.0971	1008	Fail	-

TABLE III
RELAXATION GAP OF λ IN MEDIUM-SCALE SYSTEMS

System	SDR	Iter1	Iter2	Gap(%)
case30	1.0413	1.0353	1.0349	0.61
case39	1.1159	1.0903	1.0902	2.30

TABLE IV
CONVERGENCE PERFORMANCE OF MEDIUM-SCALE SYSTEMS

System	SDR	Iter1	Iter2
case30	$1.33 \times 10{-}2$	$6.48 \times 10{-}6$	$4.48 \times 10{-}6$
case39	$1.93 \times 10{-}2$	$1.05 \times 10{-}5$	$6.69 \times 10{-}6$

From the numerical results of MLA in Table II, BARON can handle all the IEEE instances whose scales are no larger than the 39-bus one. SCIP can deal with the smallest two systems. The optimal values will serve as the baseline for solution accuracy. It can be observed from TABLE I that Algorithm 1 successfully solves all the testing systems. The MLA results are very close to those offered by BARON, validating its effectiveness. The computation time of BARON grows quickly with the increase of system scale, while the computation time of Algorithm 1 is acceptable in all instances.

For the former two systems, the initial SDP relaxation is already exact. As for the latter two systems with larger scales, additional iterations are needed to narrow the relaxation gap. The loadability value λ in each iteration is shown in TABLE III. It is observed that the objective value decreases gradually, and the initial relaxation gap is 0.61% and 2.30% for the 30-bus system and 39-bus system, respectively. The value sequence of R generated by Algorithm 1, which is used to quantify the residual of non-zero eigenvalues, is shown in TABLE IV. After two iterations, R has an order of magnitude of 10^{-6}, implying that the X matrix is almost rank-1. Indeed, if the iteration continues, R decreases very slowly, and it is very difficult to find an exact rank-1 solution. The error can be seen from the optimal values shown in TABLE I and TABLE II, which is in fact very small.

TABLE V
CONVERGENCE PERFORMANCE OF 30-BUS SYSTEM UNDER DIFFERENT PENALTY PARAMETER SETTINGS

	Penalty Growth Rate		
Initial Penalty Value	1.1	1.5	2.0
0.0016	3	2	3
0.0018	1	1	1
0.016	3	2	3
0.02	2	2	2
0.04	1	1	1
0.06	1	1	1
0.08	1	1	1
0.1	7	Fail	Fail
0.13	7	Fail	Fail
0.2	Fail	Fail	Fail
0.25	Fail	Fail	Fail

TABLE VI
CONVERGENCE PERFORMANCE OF 39-BUS SYSTEM UNDER DIFFERENT PENALTY PARAMETER SETTINGS

	Penalty Growth Rate		
Initial Penalty Value	1.1	1.5	2.0
0.0012	1	1	1
0.005	1	1	1
0.01	2	2	2
0.013	2	2	2
0.015	3	2	2
0.018	2	Fail	3
0.02	2	2	Fail
0.04	Fail	Fail	3
0.05	Fail	Fail	Fail
0.06	5	Fail	Fail
0.08	Fail	Fail	Fail

Finally, we test the convergence performance of medium scale systems with different penalty parameters (initial value and growth rate). Specifically, the total number of iterations are tested. Results of the 30-bus system and the 39-bus system are presented in Table V and Table VI, respectively. We can see that the performance is mainly influenced by the initial penalty. Algorithm 1 may fail to converge in 30 iterations if it exceeds a certain value.

IV. CONCLUSIONS

The maximum loadability problem, a fundamental problem related to power system steady-state operation and voltage stability margin, can be formulated as a special power flow optimization problem, and solved by sequential convex optimization procedure. With rank penalty terms added in the SDP relaxation model, the rank of the square matrix solution can be reduced, and thus the solution quality can be improved. Case studies show that general-purpose (local) nonlinear programming solvers encounter difficulties when facing the MLA problem, while branch-and-bound based global solvers are not scalable and efficient. The proposed method also has some shortcomings. 1) the performance is somehow sensitive to the selection of penalty parameter. 2) solving SDPs repeatedly may not be efficient for large-scale systems.

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